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A new analytical modelling for fractional telegraph equation via Laplace transform

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ABSTRACT

The main aim of the present work is to propose a new and simple algorithm for space-fractional telegraph equation, namely new fractional homotopy analysis transform method (HATM). The fractional homotopy analysis transform method is an innovative adjustment in Laplace transform algorithm (LTA) and makes the calculation much simpler. The proposed technique solves the nonlinear problems without using Adomian polynomials and He's polynomials which can be considered as a clear advantage of this new algorithm over decomposition and the homotopy perturbation transform method (HPTM). The beauty of the paper is error analysis which shows that our solution obtained by proposed method converges very rapidly to the known exact solution. The numerical solutions obtained by proposed method indicate that the approach is easy to implement and computationally very attractive. Finally, several numerical examples are given to illustrate the accuracy and stability of this method.

1. Introduction

In the last few decades, fractional calculus found many applications in various fields of physical sciences such as viscoelasticity, diffusion, control, relaxation processes and so on [1–9]. Suspension flows are traditionally modelled by parabolic partial differential equations. Sometimes they can be better modelled by hyperbolic equations such as the telegraph equation, which have parabolic asymptotics. In particular the experimental data described in [10,11] seem to be better modelled by the telegraph equation than by the heat equation. The telegraph equation is used in signal analysis for transmission and propagation of electrical signals and also used modeling reaction diffusion [12,13]. The different type solutions of the fractional telegraph equations have been discussed by Momani [14] by using decomposition method, Yildirim [15] by homotopy perturbation method, Chen et al. [16] by method of separable variables, Huang [17] by Cauchy problem, Biazar and Eslami [18] by using differential transform method, Sevimlican [19] by variational iteration method, Azab and Gamel [20] by Rothe–Wavelet–Galerkin method. Our concern in this work is to consider the space-fractional telegraph equations as

$$D_x^{2\alpha}u(x,t) = D_t^2u(x,t) + aD_t(x,t) + bu^n(x,t) + f(x,t), \quad 0 < \alpha \leqslant 1, \tag{1.1}$$

where a, b and n are given constants, $f(x, t)$ is given function.

In this paper, the homotopy analysis transform method (HATM) basically illustrates how the Laplace transform can be used to approximate the solutions of the linear and nonlinear partial differential equation by manipulating the homotopy analysis method. The proposed method is coupling of the homotopy analysis method and Laplace transform. The main advantage of this proposed method is its capability of combining two powerful methods for obtaining rapid convergent

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series for space fractional telegraph equations. Homotopy analysis method (HAM) was first proposed and applied by Liao [21–24] based on homotopy, a fundamental concept in topology and differential geometry. The HAM is based on construction of a homotopy which continuously deforms an initial guess approximation to the exact solution of the given problem. An auxiliary linear operator is chosen to construct the homotopy and an auxiliary parameter is used to control the region of convergence of the solution series. The HAM provides greater flexibility in choosing initial approximations and auxiliary linear operators and hence a complicated nonlinear problem can be transformed into an infinite number of simpler, linear sub problems, as shown by Liao and Tan [25]. The HAM has been successfully applied by many researchers for solving linear and non-linear partial differential equations [26–32]. In recent years, many authors have paid attention to studying the solutions of linear and nonlinear differential and integral equations by using various methods with combined the Laplace transform. Among these are the Laplace decomposition methods [33–34], homotopy perturbation transform method [35–39]. Some authors [40–43] have solved fractional differential equation by using different numerical techniques. Recently, Khan et al. [44] has applied to obtain the solutions of the Blasius flow equation on a semi-infinite domain by coupling of homotopy analysis and Laplace transform method.

The main objective of the present article is introduce a new analytical and approximate solution of space-fractional telegraph equation by means of fractional homotopy analysis transform method, which is coupling of homotopy analysis method and Laplace transform method.

2. Basic definition of fractional calculus and Laplace transform

In this section, we give some basic definitions and properties of fractional calculus theory which shall be used in this paper:

Definition 2.1. A real function $f(t),\ t>0$ is said to be in the space $C_\mu,\ \mu\in R$ if there exists a real number $p>\mu$, such that $f(t)=t^p f_1(t)$ where $f_1(t)\in C(0,\infty)$ and it is said to be in the space C_n if and only if $f^{(n)}\in C_\mu,\ n\in N$.

Definition 2.2. The left sided Riemann-Liouville fractional integral operator of order $\mu\geqslant 0$, of a function $f\in C_\alpha,\ \alpha\geqslant -1$ is defined as [45,46]

$$I^\mu f(t)=\begin{cases}\frac{1}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}f(\tau)d\tau, & \mu>0,\ t>0,\\ f(t), & \mu=0\end{cases} \tag{2.1}$$

where $\Gamma(.)$ is the well-known Gamma function.

Definition 2.3. The left sided Caputo fractional derivative of $f,\ f\in C_{-1}^m,\ m\in N\cup\{0\}$ is defined as [5,47]

$$D_*^\mu f(t)=\frac{\partial^\mu f(t)}{\partial t^\mu}=\begin{cases}I^{m-\mu}\left[\frac{\partial^m f(t)}{\partial t^m}\right], & m-1<\mu<m,\ m\in N,\\ \frac{\partial^m f(t)}{\partial t^m}, & \mu=m,\end{cases} \tag{2.2}$$

Note that [5,47]

(i) $I_t^\mu f(x,t)=\dfrac{1}{\Gamma(\mu)}\displaystyle\int_0^t \frac{f(x,t)}{(t-s)^{1-\mu}}dt,\quad \mu>0,\ t>0,$

(ii) $D_*^\mu f(x,t)=I_t^{m-\mu}\dfrac{\partial^m f(x,t)}{\partial t^m}\quad m-1<\mu\leqslant m,$

Definition 2.4. The Laplace transform of continuous (or an almost piecewise continuous) function $f(t)$ in $[0,\infty)$ is defined as

$$F(s)=L[f(t)]=\int_0^\infty e^{-st}f(t)dt. \tag{2.3}$$

where s is real or complex number.

Definition 2.5. The Laplace transform of $f(t)=t^\alpha$ is defined as [5]:

$$L[t^\alpha]=\int_0^\infty e^{-st}t^\alpha dt.=\frac{\Gamma(\alpha+1)}{s^{(\alpha+1)}},\quad R(s)>0,\quad R(\alpha)>0, \tag{2.4}$$

Definition 2.6. The Laplace transform $L[f(t)]$ of the Riemann–Liouville fractional integral is defined as [5]:

$$L[I^\alpha f(t)]=s^{-\alpha}F(s). \tag{2.5}$$