Applied Mathematical Modelling 39 (2015) 6767–6779

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Numerical solutions of systems of high-order Fredholm integro-differential equations using Euler polynomials

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ARTICLE INFO

Article history:
Received 24 February 2013
Received in revised form 31 January 2015
Accepted 12 February 2015
Available online 4 March 2015

Keywords:
Linear FIDEs
Error estimation
Euler polynomials

ABSTRACT

In this paper, a novel method called Euler collocation method is presented to obtain an approximate solution for systems of high-order Fredholm integro-differential equations. The most significant features of this method are its simplicity and excellent accuracy. After implementation of our method, the main problem would be transformed into a system of algebraic equations such that its solutions are the unknown Euler coefficients. In addition, under several mild conditions the error and stability analysis of the proposed method are discussed. Finally, complete comparisons with other methods and superior results confirm the validity and applicability of the presented method.

1. Introduction

Some important problems in science and engineering can usually be reduced to a system of integral and integro-differential equations. Integro-differential equation has attracted much attention and solving this equation has been one of the interesting tasks for mathematicians. In this research we try to introduce a solution of a system of high-order linear Fredholm integro-differential equations (FIDEs) with variable coefficients in the form

$$\sum_{n=0}^{m}\sum_{j=1}^{k}U_{i,j}^{n}(x)y_j^{(n)}(x)=g_i(x)+\int_a^b\sum_{j=1}^{k}K_{i,j}(x,t)y_j(t)dt,\quad i=1,2,\ldots,k,\ 0\leqslant a\leqslant x\leqslant b, \tag{1}$$

with the mixed conditions

$$\sum_{j=0}^{m-1}\left(a_{i,j}^{n}y_n^{(j)}(a)+b_{i,j}^{n}y_n^{(j)}(b)\right)=\mu_{n,i},\quad i=0,1,\ldots,m-1,\ n=1,\ldots,k, \tag{2}$$

where $y_j^{(0)}(x)=y_j(x)$ is an unknown function. Also, $U_{i,j}^{n}(x), g_i(x)$ and $K_{ij}(x,t)$ are continuous functions defined on the interval $a\leqslant x,t\leqslant b$. Moreover, the functions $K_{i,j}(x,t)$ for $i,j=1,2,\ldots,k$ can be expanded Maclaurin series and also $a_{i,j}^{n}, b_{i,j}^{n}$ and $\mu_{n,i}$ are appropriate constants. Many researchers are shown in solving such types of integro-differential equations system such as, Adomian decomposition method [1], the Tau method [2], Galerkin method [3], He's Homotopy perturbation method [4], differential Transform method [5] and rationalized Haar functions method [6].

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http://dx.doi.org/10.1016/j.apm.2015.02.022
0307-904X/

Operational matrices of differentiation and integration have become increasingly important in the field of numerical solution of FIDEs. As some primary research works, one can refer to the Taylor, Chebyshev and Legendre matrix methods which have been used by *Sezer* et al. [7–9] to solve linear differential, Fredholm–Volterra integro-differential equations and their systems. Recently, Yüzbaşı et al. [10–14] have studied the Bessel matrix and collocation methods for numerical solutions of the neutral delay differential equations, the pantograph equations, the Lane–Emden differential equations, Fredholm integro-differential equations and Volrerra integral and Fredholm integro-differential equation systems.

In this study, the basic ideas of the previous works are developed and we introduce a new method called the Euler approximation technique to find an approximate solution of (1) expressed in the truncated Euler series form

$$y_i(x) = \sum_{n=0}^{N} c_{i,n} E_n(x), \quad i = 1,2,\ldots,k, \quad 0 \leqslant a \leqslant x \leqslant b, \tag{3}$$

so that $c_{i,n}$ for $i = 1,2,\ldots,k$ and $n = 0,1,\ldots,N$ are the unknown Euler coefficients, and $E_n(x)$ for $n = 0,1,\ldots,N$ are the Euler polynomials of the first kind which are constructed from the following relation

$$\sum_{k=0}^{n} \binom{n}{k} E_k(x) + E_n(x) = 2x^n, \tag{4}$$

with $E_0(x) = 1$, where $\binom{n}{k}$ is a binomial coefficient. Explicitly, the first basic polynomials are expressed by $n = 1;\ E_1(x) = x - \frac{1}{2},\ n = 2;\ E_2(x) = x^2 - x$ and if $n = 3;\ E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}x$.

2. Preliminaries and notations

In this section, we state some basic results about polynomial approximations. These important properties will enable us to solve the systems of Fredholm integro-differential equations.

Definition 1 [15]. For a given continuous function $f \in C[a,b]$ a best approximation polynomial of degree N is a polynomial $p \in P_n$ such that

$$\|f - p\|_\infty \leqslant \|f - q\|_\infty, \quad \forall q \in P_N$$

where P_N is the $(n+1)$-dimensional subspace of $C[a,b]$ spanned by the functions $1, x, \ldots, x^N$ and the uniform norm is defined by $\|f\|_\infty = \max_{a \leqslant x \leqslant b} \mid f(x) \mid$.

Theorem 1 [16]. *Given $N+1$ distinct nodes $x_0, x_1, \ldots, x_N$ and $N+1$ corresponding values $f_0, f_1, \ldots, f_N$ then there exists a unique polynomial $p_N \in P_N$ such that $p_N(x_i) = f_i$ for $i = 0,1,\ldots,N$. p_N is called the interpolating polynomial of f.*

The best approximation polynomials p is also an interpolant of f at $N+1$ nodes and the error is given by [17]:

$$\|f - p_N\|_\infty \leqslant (1 + \Gamma_N(X))\|f - p\|_\infty \leqslant 6(1 + \Gamma_N(X))\omega\left(\frac{b-a}{2N}\right),$$

where $\Gamma_N(X)$ denotes the Lebesgue constant

$$\Gamma_N(X) = \left\| \sum_{i=0}^{n} \left| l_i^{(X)}(x) \right| \right\|_\infty, \tag{5}$$

where $l_i^{(X)}(x) \in P_N$ is the ith Lagrange cardinal polynomial associated with the grid $X = (x_i)_{i=0,1,\ldots,N}$ and ω is the modulus of continuity of f.

Now, we recall some results on the Euler polynomials [18–20], which play important roles in the proposed collocation scheme. The Euler polynomials $E_n(x)(n = 0,1,\ldots)$ satisfy the following formula

$$E_n(x) = \frac{1}{n+1} \sum_{k=1}^{n+1} (2 - 2^{k+1}) \binom{n+1}{k} B_k(0) x^{n+1-k},$$