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Solution of the one-phase inverse Stefan problem by using the homotopy analysis method

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ABSTRACT

In this paper we consider the one-phase inverse Stefan problem consisting in determination of the temperature distribution in given domain together with the temperature and the heat flux on one of boundaries of the region. For solving this problem the homotopy analysis method will be used. Concept of this method lies in forming the series, elements of which satisfy some differential equation. It is proven in the paper that if this series is convergent then its sum represents the solution of considered equation. Sufficient condition of this convergence is given in this elaboration, as well as the estimation of error of the solution approximated by taking the partial sum of the above mentioned series. Application of the method is illustrated by examples.

1. Introduction

The homotopy analysis method [1–5], developed by Shijun Liao, serves for solving the operator equations (especially the nonlinear equations). The method has already found a number of applications in solving many problems, such that the nonlinear Cauchy problem of parabolic–hyperbolic type [6], differential-difference equations (Toda lattice system) [7], nonlinear reaction–diffusion–convection problems [8], nonlocal initial boundary value problem [9], fractional differential equations [10–12], Fredholm and Volterra integral equations [13]. The homotopy analysis method has been also used for investigating the heat conduction problems [14–20], whereas in papers [21,22] the method has been applied for solving the inverse heat conduction problem. Theoretical results concerning, among others, convergence of the method are included, for example, in works [3,4,23–30].

Only in few simple cases it is possible to find an exact solution of the inverse Stefan problem. In other cases the only thing possible to obtain is the approximate solution (see for example [31–38]). Johansson and the co-authors [39–41] have described how to apply the method of fundamental solutions for solving the inverse Stefan problem. Whereas Liu [42] for solving two typical inverse Stefan problems has used the Lie-group shooting method. In papers [43,44] there is presented the application of Adomian decomposition method combined with optimization techniques for solving the one-phase inverse Stefan problem with boundary condition of the first or second kind. Next, in papers [45–47] the application of variational iteration method and homotopy perturbation method for solution of the inverse Stefan problem is discussed.

In paper [48] there is shown a usage of the homotopy analysis method for solving the linear and nonlinear integral equations of the second kind. This paper presents also the proof of convergence of the method in case of discussed integral equations as well as the estimation of error of approximate solution. In the current paper we deal with applying the

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homotopy analysis method for solving the inverse problem described by means of the heat conduction differential equation, which is the first attempt for considering this problem with the aid of HAM method. Investigated one-phase inverse Stefan problem consists in determination of the temperature distribution in given domain together with the temperature and the heat flux on one of boundaries of the region. Additional information in the inverse problem is given by the location of moving interface, which means that the considered problem is the, so called, design problem. Idea of the proposed method lies in construction of a series, elements of which satisfy some differential equation. We prove in the paper that if the formed series is convergent then sum of this series gives the solution of considered equation. Moreover, we present the sufficient condition of this convergence. Error of approximate solution obtained by considering the partial sum of the series is given as well. Examples illustrating application of discussed method are also presented. Additionally, we compare the investigated method with other methods of similar type (homotopy perturbation method, Adomian decomposition method and variational iteration method).

2. Statement of the problem

Investigated problem is defined in region $D = \{(x,t);\ t \in [0,t^*),\ x \in [0,\xi(t)]\}$. On boundary of this domain three components are distributed

$$\Gamma_0 = \{(x,0); \quad x \in [0, s = \xi(0)]\}, \tag{1}$$

$$\Gamma_1 = \{(0,t); \quad t \in (0,t^*)\}, \tag{2}$$

$$\Gamma_g = \{(x,t); \quad t \in (0,t^*), \quad x = \xi(t)\}, \tag{3}$$

where function ξ describes position of the moving interface.

In domain D we consider the heat conduction equation

$$\alpha\,\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}, \tag{4}$$

where α denotes the thermal diffusivity, and u, t and x refer to temperature, time and spatial location, respectively. On boundary Γ_0 the initial condition is given

$$u(x,0) = \varphi(x). \tag{5}$$

Whereas on the moving interface Γ_g the condition of temperature continuity and the Stefan condition [49,50] are defined

$$u(\xi(t),t) = u^*, \tag{6}$$

$$-k\,\frac{\partial u(x,t)}{\partial x}\Big|_{x=\xi(t)} = \kappa\,\frac{d\xi(t)}{dt}, \tag{7}$$

where k denotes the thermal conductivity, κ is the latent heat of fusion per unit volume and u^* is the phase change temperature.

Discussed inverse Stefan problem consists in finding function u, describing the temperature distribution in domain D, as well as the functions θ and q describing the temperature and the heat flux on boundary Γ_1:

$$u(0,t) = \theta(t), \tag{8}$$

$$-k\,\frac{\partial u(x,t)}{\partial x}\Big|_{x=0} = q(t), \tag{9}$$

such that the sought functions would satisfy Eqs. (4)–(7). All other functions (φ, ξ) and parameters (α, k, κ, u^*) are known.

3. Foundations of the homotopy analysis method

Using homotopy analysis method we are able to solve the following operator equation

$$N(u(z)) = 0, \quad z \in \Omega, \tag{10}$$

where N denotes the operator and u express the sought function.

The procedure is began by defining the, so called, zero-order deformation equation

$$(1-p)\,L(\Phi(z;p) - u_0(z)) = p\,h\,H(z)\,N(\Phi(z;p)), \tag{11}$$

where $p \in [0,1]$ denotes the embedding parameter, L describes the auxiliary linear operator with property $L(0) = 0, h \neq 0$ is the auxiliary parameter, $H(z) \neq 0$ is the auxiliary function and u_0 represents the initial approximation of sought solution.