Applied Mathematical Modelling 39 (2015) 6860–6879

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Elasticity-based beam vibrations for various support conditions

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ARTICLE INFO

Article history:
Received 18 November 2013
Received in revised form 20 December 2014
Accepted 10 February 2015
Available online 6 March 2015

Keywords:
Beam
Displacement
Ritz
Timoshenko
Rayleigh
Euler

ABSTRACT

The accuracy of common beam theories in determining resonant frequencies and mode shapes for various support conditions is investigated in this paper. This study analyzed the influence of three assumptions employed in the Euler–Bernoulli, Rayleigh, and Timoshenko beam theories by comparing their results to a three-dimensional, elasticity-based approximation for three different support cases: fixed-free, fixed–fixed, and simply-supported. Each of the theories of interest were applied to both isotropic and orthotropic beams of varying length and compared with elasticity-based solutions to study the influence of slenderness ratio and anisotropic properties. The discrepancies caused by each of these effects are discussed and suggestions for the applicability of the common beam theories are provided.

1. Introduction

The vibrating beam is a well studied mechanics problem with a multitude of practical applications in engineering [1–17]. These applications including the beam as standard structural elements [18] to more recent applications of beam configurations as sensors and actuators [19]. Problems involving a vibrating beam are typically approached using one of the common beam representations: the Euler–Bernoulli, Rayleigh, and Timoshenko theories. The use of these theories provides reasonable results for many engineering applications; however, all three rely upon restrictive kinematic assumptions. A main component that is missing from the collection of research on the vibrating beam is an investigation into when these assumptions begin to limit accuracy when compared with results obtained using elasticity theory with less restrictive constraints on the displacement fields. In order to confidently use these beam theories, their limitations should be well understood.

In this paper a fourth method, which can be referred to as three-dimensional Ritz elasticity (3DRE) beam theory, to analyze vibrational beam mechanics is introduced that better approximates a fully three-dimensional elasticity theory. This approach will be employed to study the accuracy of the common beam theories and the point at which they fail to be acceptable analysis tools. When studying the effectiveness of common beam theories, three different boundary condition cases will be evaluated: fixed-free, fixed–fixed, and simply supported. Another common support condition in beam theory is the propped cantilever, but the modes for this beam are the anti-symmetric modes for the fixed–fixed beam [18] and hence not specifically considered in this study. It is extremely common in engineering practice to use beam frequencies that have been tabulated by various researchers for each of these boundary conditions using the simplified Euler–Bernoulli theory. These are the most widely available results and are a function of only beam length, density, elastic modulus, and beam

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http://dx.doi.org/10.1016/j.apm.2015.02.023

cross-section. The intent of this study is to quantify levels of accuracy as the beam becomes stocky and the ratio between the elastic and shear moduli increases.

The primary contribution of this study is to provide quantitative limits of typically used beam theories compared with results from an elasticity model that does not possess the strict kinematic limitations inherent in one-dimensional representations of flexural beam deformation. Initial guidelines are also presented related to the influence of elasticity-based imposition of beam supports as opposed to the same supports modeled using beam theories.

2. Governing equations

This section will present the beam theories which were studied including the equations which govern each of these models. In accordance with common practice for beam theory discussion, dimensionless variables will be employed for the geometric and vibrational parameters. Also, a consistent coordinate system will be used for all of the calculations where the x axis is parallel to the width of the beam, the y axis provides the vertical dimension of the beam, and the z axis is used for the axial dimension of the beam. The variables u, v, and w will be used to represent the displacement along each of these axes, respectively. These are shown in Fig. 1.

2.1. Euler–Bernoulli beam theory

The Euler–Bernoulli beam theory, also known as the classical beam theory, Euler beam theory, Bernoulli beam theory, or Bernoulli–Euler beam theory, is the most commonly used theory. It is very simple and provides acceptable results for many engineering problems. This theory includes both the strain energy from bending and the kinetic energy from transverse displacement, but does not consider the effects of rotary inertia or shear displacement. Ignoring these two contributions leads to overestimates of the natural frequencies of a vibrating beam, especially for higher modes of vibration and less slender beams when these two factors are not negligible [18].

The governing differential equation of motion for the Euler–Bernoulli model is given by [18]

$$\rho A \frac{\partial^2 v(z,t)}{\partial t^2} + \frac{\partial^4 v(z,t)}{\partial z^4} = f(z,t), \tag{1}$$

where ρ is the density of the material and A is the cross-sectional area of the beam. Only free vibration is considered in this paper, so the forcing function, $f(z,t)$, is set to zero. To simplify the analysis the principle of separation of variables will be used to isolate an equation for transverse displacement that is dependent on the axial direction only and not time [18]. This new equation is referred to as the spatial solution and it is of the form

$$v_{EB}(z) = C_1 \sin az + C_2 \cos az + C_3 \sinh az + C_4 \cosh az. \tag{2}$$

In this equation, the parameter a is the dimensionless wave number [18] and the EB subscript has been added to the transverse displacement to distinguish this variable from those of other beam theories. For the Euler–Bernoulli theory the wave number is given by

$$a = \rho A \omega^2, \tag{3}$$

where ω is the natural frequency of the beam. Using the spatial solution above along with appropriate boundary conditions for each of the support cases, a system of four equations results. The solution of this system of equations provides the frequency equation for each specific case.

2.2. Rayleigh beam theory

The Rayleigh beam theory includes a correction that allows for rotary inertia in the beam [1]. The addition of this term provides a slight improvement over the Euler–Bernoulli model by slightly reducing the overestimation of the natural

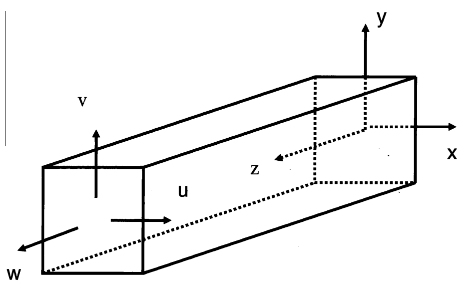

Fig. 1. The coordinate axes (x,y,z) for the general beam and the displacement components (u,v,w) in the respective directions.