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# Dynamic stability of nonlocal Voigt–Kelvin viscoelastic Rayleigh beams

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ABSTRACT

The dynamic stability problem of a viscoelastic nanobeam subjected to compressive axial loading, where rotary inertia is taken into account, is investigated. The paper is concerned with the stochastic parametric vibrations of a Voigt–Kelvin nanobeam based on Eringen's nonlocal elasticity theory of the Helmholtz and bi-Helmholtz type of kernel. The axial force consists of a constant part and a time-dependent stochastic function. By using the direct Liapunov method, bounds of the almost sure asymptotic stability of a viscoelastic nanobeam are obtained as a function of retardation time, variance of the stochastic force, geometric ratio, scale coefficients, and intensity of the deterministic component of axial loading. Numerical calculations were done for the Gaussian and harmonic process. When the excitation is a real noise process, the advanced numerical simulation based on the Monte Carlo method is presented for moment Liapunov exponents numerical determination.



## 1. Introduction

A simple model of a one-dimensional continuous structure made from nano materials and of nanometer dimension is referred to as a nanobeam. Recently, it has been extensively utilized as nanostructure components for nanoelectromechanical and microelectromechanical systems. The dynamic problems of single beams based on various theories have been studied by many researchers.

Applications of the nonlocal continuum theory to nanotechnology were initially addressed by Peddieson et al. [1], who analyzed the static deformations of beam structures based on the simplified nonlocal Eringen's theory [2]. Lu et al. [3] used the nonlocal Euler–Bernoulli and Timoshenko beam theory to study wave and vibration properties of the single- and double-walled nanotubes. By using various nonlocal beam theories, analytical solutions of bending, vibration and buckling were presented by Reddy [4]. The natural frequencies of the bending vibrations of a nanocantilever with linearly changed cross-section were obtained by Aranda-Ruiz et al. [5]. A nonlocal viscoelastic constitutive model and external velocity-dependent damping model to analyze the dynamic characteristics of Timoshenko beams with different boundary conditions using the transfer function method was considered by Lei et al. [6]. Based on Rayleigh beam theory and by using the finite element method, Chang et al. [7] derived the equations of motion of an axially moving beam. Floquet theory was employed to investigate the effect of the axial-movement frequency on instantaneous natural frequencies and the stability of a telescopically moving beam with time-dependent velocity. Size dependent behavior of electrostatically-actuated nano-beam

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considering vdW and Casimir forces was investigated by Mobki et al. [8]. The pull-in voltage, detachment length and natural frequency were compared in both classic and modified couple stress theories for aluminum nano-beams.

By using Euler–Bernoulli and Timoshenko beam theories, dynamic instability of nanobeams subjected to time dependent stochastic loading was treated by Tylikowski [9,10]. Direct Liapunov method is used for double-beam system dynamic stability and instability analysis [11]. Influence of rotary inertia is investigated for viscoelastic symmetric cross-ply laminated plates in [12], and rotating shafts in [13]. Asymptotic stability and almost sure asymptotic stability of a beam, taking into account the effect of the nonlocal elasticity and damping and using the method of maximal Liapunov exponent, was considered by Potapov [14].

The purpose of the present paper is to investigate the almost sure stability of a viscoelastic nanobeam as a function of retardation time, variance of the stochastic force, geometric ratio, scale coefficient, and intensity of the deterministic component of axial loading. The principal contribution of this paper is to clearly fix the boundaries of stability regions when the influence of rotary inertia of the viscoelastic nanobeam is taken into account.

The present paper is organized as follows. According to the tensor notation, the nonlocal constitutive relations are given in Section 2. A partial differential equation of transverse motion of a viscoelastic nanobeam based on Eringen's nonlocal elasticity theory for the Helmholtz and bi-Helmholtz type of kernel and Rayleigh beam theory is derived in Section 3. For the governing differential equation of nanobeam, the definition of almost-sure stability problem is given in Section 4. For non-white excitation by using Liapunov functional method, the conditions of almost-sure stability are obtained in Section 5. When the viscoelastic nanobeam is subjected to real noise, in Section 6 stability analysis is performed by using the proposed developed simulation based on the Monte–Carlo method. The numerical procedure of determining the boundaries of stability, as well as the analysis of obtained results, is given in Section 7. Section 8 ends the paper with concluding remarks.

Contrary to papers [11–13], in this work the direct Liapunov method is applied in the nonlocal theory. Also, papers [12,13] give the influence of rotary inertia in the function of the viscous damping coefficient for a Gaussian and harmonic process, while here the same influence is given in the function of retardation time. Additionally, the analysis is presented for a wideband real-noise process.

## 2. Nonlocal constitutive relations

The theory of Eringen [2] nonlocal elasticity in the integral formulation assumes that the stress at a given reference point $\mathbf{X}$ is a function of the strain at all points $\mathbf{X}'$ in the body, through a weighting kernel $\alpha(|\mathbf{X}'-\mathbf{X}|)$

$$t_{ij}(\mathbf{X}) = \int_V \alpha\left(|\mathbf{X}'-\mathbf{X}|\right)\sigma_{ij}(\mathbf{X}')\,dV, \tag{1}$$

where $t_{ij}$ and $\sigma_{ij}$ are the nonlocal and local (classical) stress tensors, respectively. Eringen proposed the weighting kernel as a Green function of a linear differential operator $\mathcal{L}$ as

$$\mathcal{L}\alpha\left(|\mathbf{X}'-\mathbf{X}|\right) = \delta\left(|\mathbf{X}'-\mathbf{X}|\right), \tag{2}$$

where $\delta$ is the Dirac function. After applying Eq. (2) to Eq. (1), the integral forms of the nonlocal stress tensor are reduced to the differential one

$$\mathcal{L}t_{ij} = \sigma_{ij}. \tag{3}$$

In Eringen's [2] nonlocal elasticity of the Helmholtz and bi-Helmholtz type, linear operators $\mathcal{L}$ are, respectively

$$\mathcal{L}_H = 1-\bar{\mu}_0^2\nabla^2, \quad \mathcal{L}_{bH} = \left(1-\bar{\mu}_1^2\nabla^2\right)\left(1-\bar{\mu}_2^2\nabla^2\right), \tag{4}$$

where $\nabla^2$ is the Laplacian operator, and $\bar{\mu}_0$ $\bar{\mu}_1$ and $\bar{\mu}_2$ are the nonnegative parameters of nonlocality. A detailed analysis of nonlocal elasticity of the bi-Helmholtz type is given by Lazar et al. [15].

## 3. Nonlocal Rayleigh beam theory

Fig. 1 shows a uniform beam of length $L$ subjected to transverse loading per unit length ($q_1$on upper side and $q_2$ on lower one), and the axial compressive load $H$. $X$-axis is placed on beam axes, and transversal loadings are parallel with $Z$-axis. A typical beam element is also shown in the figure (without the forces due to viscous damping).

The dynamic equilibrium of the element gives

$$\begin{aligned} &\rho A\frac{\partial^2 W}{\partial T^2} + c_1\frac{\partial W}{\partial T} = \frac{\partial V}{\partial X} + q, \\ &\rho I\frac{\partial^2 \psi}{\partial T^2} + c_2\frac{\partial \psi}{\partial T} = V - \frac{\partial M}{\partial X} + H\frac{\partial W}{\partial X}, \end{aligned} \tag{5}$$