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Computing parallel curves on parametric surfaces

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ABSTRACT

Generating parallel curves on parametric surfaces is an important issue in many industrial settings. Given an initial curve (called the base curve or generator) on a parametric surface, the goal is to obtain curves *on* the surface that are parallel to the generator. By *parallel curves* we mean curves that are at a given distance from the generator, where distance is measured point-wise along certain characteristic curves (on the surface) orthogonal to the generator. Except for a few particular cases, computing these parallel curves is a very difficult and challenging problem. In fact, only partial, incomplete solutions have been reported so far in the literature. In this paper we introduce a simple yet efficient method to fill this gap. In clear contrast with other existing techniques, the most important feature of our method is its *generality*: it can be successfully applied to any differentiable parametric surface and to any kind of characteristic curves on surfaces. To evaluate our proposal, some illustrative examples (not addressed with previous methods) for the cases of section, vector-field, and geodesic parallels are discussed. Our experimental results show the excellent performance of the method even for the complex case of NURBS surfaces.

1. Introduction

A classical academic and industrial problem is the determination of parallel curves on parametric surfaces. This problem appears recurrently in many industrial processes. Parallel curves are widely used, for instance, in manufacturing, for toolpath generation in sculptured surface machining. Also in rapid prototyping, to fabricate additively a solid object or assembly from CAD models by using 3D printing technologies such as laser sintering, stereolithography, and laminated object manufacturing. Parallel curves are also intensively used in many other fields, such as metrology and quality control assessment of final manufactured products.

The problem of computing parallel curves on parametric surfaces can be stated as follows: given an initial curve $\mathbf{C}$ (that will be called onwards the *base curve* or *generator*) on a parametric surface $\mathbf{S}$, the goal is to obtain curves on that surface that are parallel to $\mathbf{C}$. By *parallel curves* we mean curves that are at a given distance point-wise from $\mathbf{C}$, where the distance is measured on $\mathbf{S}$ along certain families of curves (called *characteristic curves* henceforth; see Section 2.1 for details) orthogonal to $\mathbf{C}$.

The problem of computing parallel curves is related to that of computing offsets of surfaces of objects represented as polygonal meshes. The latter has been the subject of intensive research, and several approaches can be found in the literature (see, for instance, [1,2]). This is largely due to its remarkable applications to surface reconstruction from sets of input data points in fields such as reverse engineering and the fact that polygonal meshes are the simplest and most common representation of 3D models. In comparison, the problem of computing parallel curves on parametric surfaces has received relatively little attention from the scientific community so far. Notable exceptions are the papers in [3,4], where the authors

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addressed the problem of constructing a particular type of parallel curves, the so-called *geodesic offsets* or *geodesic parallels*, defined as the locus of points at constant distances measured from a curve on a surface along geodesic curves drawn orthogonal to that curve. Unfortunately, both approaches are extremely limited in scope. The work in [3] presents an algorithm for efficient tracking of the geodesic parallels on surfaces. The method works well but is severely restricted to the particular case of surfaces of revolution and is not applicable to other types of parametric surfaces. The authors in [4] provided an algorithm to compute such geodesic offsets on free-form NURBS surfaces. The method performs well but is strictly focused on geodesic curves and no other type of curves are supported.

Since then, to the best of our knowledge, no other attempt has been made to address this issue in the context of geometric processing or manufacturing. In other words, the existing methods reported in the literature so far provide suitable solutions for some particular cases of curves and/or surfaces but never address the problem in all its generality. Consequently, there is a lack of a unified methodology to compute parallel curves for any family of characteristic curves on general parametric surfaces. The present work is aimed at filling this gap.

1.1. Motivation of this work

Our motivation to construct parallel curves on parametric surfaces comes mainly from the field of sculptured surface machining, a point-based CNC (computer numerically controlled)-milling process where a sequence of cutter-contact points are traced by milling cutters by following a pattern of tracing or scanning usually called *tool-path topology* [5]. According to [6], those tool-path patterns can be grouped into four types: serial-pattern, radial-pattern, strip-pattern and contour-pattern. The first two groups, intended for machining an area, consist of trajectories on surfaces that are (either locally or globally) at given distances from a generator. For instance, the serial-pattern include strategies such as BC-parallel and BC-normal (BC stands for boundary curve) that typically require the computation of curves on surfaces that are parallel (or normal) to a prescribed boundary curve. The radial-pattern includes the case of contour-parallel offsets, used for machining an area or a pocket. Other strategies such as strip-parallel topology are used to remove strips of uncut-regions that typically appear in finish-machining with ball-end cutters along the sharp concave fillets [6] by employing small-size cutters along unconnected parallel trajectories on the strip. Another example of application of parallel curves is for tool-path generation procedures for pocketing, the most typical roughing operation for die-cavity machining. In this case, contour-parallel curves for flat-end or round-end cutter milling are used. Such curves can be regarded as parallel curves to the boundary-pocketing curve at given distances measured on the design-surface.

A reason why parallel curves are so often used in CNC-machining is to ensure that the space between adjacent tool-paths is kept constant in either the three-dimensional space (i.e., on the surface) or in the surface parametric domain. As pointed out in [7, pp. 286], geodesic parallels have been applied to tool-path generation in zig-zag finishing with 3-axis machining and ball-end cutter so that the scallop-height (the cusp height of the material removed by the cutter) becomes constant [8,9], thus optimizing the size of the cutter location data and consequently reducing the machining time. Recent papers [10–12] have pointed out, however, that under certain conditions the scallop-height is larger than originally expected and proposed methods to improve the process by achieving fewer and shorter tool-paths. Another recent paper [13] focuses on this problem and introduces a new approach for generating constant cusp height tool-paths by using a new metric, referred to as cusp-metric, defined from the curvature tensors of a workpiece and a tool surface and then constructing geodesic parallels on the resulting Riemannian manifold. All these works emphasized the difficulties of the constant scallop-height problem and the importance of improving current solutions for better performance and efficiency. Some of recent approaches consider the use of parallel curves (mostly geodesic parallels so far) as a suitable tool for further improvement.

A striking remark is that tool-path topology planning has been largely seen as a plain *distance minimization problem*, without taking care of the subtle details of technological "machinery" for CNC milling. From this point of view, geodesic curves are the preferred (often the must-use) mathematical tools as they exhibit the much desired minimum-distance property. But efficient strategies are not that easy; there are always many technological issues to be taken into account (machining time, surface quality, process stability, gouge avoidance, collision prevention and so on) so a balanced trade off among the various factors is usually required. This opens the door to alternative strategies where other kinds of parallel curves could be advantageous for some specialized tasks. An illuminating example can be found in [14], where gradient curves have already been mentioned as useful tools for some manufacturing problems. The present contribution is aimed at providing potential users with a general methodology to generate different collections of parallel curves (including but not restricted to geodesic parallels) on arbitrary smooth parametric surfaces and, therefore, enriching current procedures with new alternative strategies.

1.2. Aim and structure of the paper

In this paper we present a simple yet efficient method to compute parallel curves on parametric surfaces. Our approach relies on geometric-differential arguments to formulate the problem as an initial value problem (IVP) of systems of ordinary differential equations (ODEs), which can be integrated through step-by-step numerical methods [15–17]. A clear advantage of our method is its generality: it can be successfully applied to any type of parametric surfaces, no matter if they are polynomial, rational or other. For a proper formulation of the method, we only require the surfaces to be differentiable. In particular, it can readily be applied to free-form parametric surfaces such as B-splines and NURBS, by far the most important geometric entities in CAD/CAM. Similarly, the method can be applied to any kind of characteristic curves on surfaces,