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Boundary equations in the finite transfer method for solving differential equation systems

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ABSTRACT

The finite transfer method is going to be used to solve a p system of linear ordinary differential equations. The complete problem is extended by adding the p boundary equations involved. It is chosen a fourth order scheme to obtain finite transfer expressions. A recurrence strategy is used in these equations and permits one to relate different points in the domain where boundary equations are defined. Finally a $2p$ algebraic system of equations is noted and solved. To show the efficiency and accuracy, the method is applied to determine the structural behavior of a bending beam with different supports and to solve a differential equation of second degree with different boundary conditions.

1. Introduction

Analytical or numerical strategies could be chosen to solve the problem of a system of linear ordinary differential equations with boundary conditions. Not always is possible to use exact methods, different numerical procedures have been arised to [1]. Several numerical methods, in last decades, have resorted to solve these boundary value problems; for example see, the shooting method [2], finite differences [3], finite element analysis [4] and the boundary element [5] methods. Other approximations to solve these problems are by the differential quadrature [6] and the finite transfer method [7,8].

Authors have employed the differential quadrature method to solve the classical beam theory [9], and the finite transfer method for the behavior of curved beams [10]. There are many authors who have studied the model of arbitrary curved beam elements [11,12]. Traditionally, when applying the Euler–Bernuolli and Timoshenko theories, the laws that govern the mechanical behavior of a curved beam are defined by equilibrium and kinematics [13,14] or dynamic equations [15]. Some authors represent this problem by means of compact energy equations [16–18]. These approximations have permitted to obtain accurate results for some types of beams: as for example, a circular arch loaded in plane [19–23] and loaded perpendicular to its plane [24], elliptical and parabolic beams loaded in plane [25–27] or a helix loaded uniformly [28].

To obtain an incremental transfer matrix equation, the finite transfer method is followed and applied to a system of differential equations [29]. First and fourth order approximations are adopted in this article but other approximations could be suitable. If we use the preceding finite expression as a recurrence scheme, both ends of the domain are related, yielding a system of algebraic equations with constant dimension p regardless of the number of intervals. In this paper, the establish-

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ment of the problem is fulfilled when the p boundary equations are added. Finally the system of $2p$ order is expressed and solved. Values at any point of the domain can be obtained, when values at the initial point are known.

The authors apply the finite transfer method incorporating the support equations on a bending beam. Besides, another example of a differential equation of second degree found in the literature [2] is solved and compared for different boundary conditions.

2. The differential problem

Let's note the system of p ODE of first order, which express the differential problem to be solved:

$$\begin{array}{ccccccccc} \frac{dy_1}{dx}+a_{11}y_1 & + & a_{12}y_2 & + & \dots & + & a_{1p}y_p & = & b_1 \\ a_{21}y_1 & + & \frac{dy_2}{dx}+a_{22}y_2 & + & \dots & + & a_{2p}y_p & = & b_2 \\ \vdots & & & & \ddots & & \vdots & & \vdots \\ a_{p1}y_1 & + & a_{p2}y_2 & + & \dots & + & \frac{dy_p}{dx}+a_{pp}y_p & = & b_p \end{array}. \tag{1}$$

In vector notation it can be annotated as:

$$\frac{d\mathbf{y}(x)}{dx} = [\mathbf{A_D}(x)]\mathbf{y}(x) + \mathbf{b_D}(x), \tag{2}$$

where,

$\mathbf{y}(x) = \{y_1(x), y_2(x), \dots, y_p(x)\}^T$ is the vector of the unknown functions,

$[\mathbf{A_D}(x)] = -\begin{bmatrix} a_{11}(x) & a_{12}(x) & \dots & a_{1p}(x) \\ a_{21}(x) & a_{22}(x) & \dots & a_{2p}(x) \\ \vdots & & \ddots & \vdots \\ a_{p1}(x) & a_{p2}(x) & \dots & a_{pp}(x) \end{bmatrix}$ is the matrix of variable coefficients and $\mathbf{b_D}(x) = \{b_1(x), b_2(x), \dots, b_p(x)\}^T$ is the independent vector term.

2.1. Exact analytical solution

The exact analytical solution of the differential system (Eq. (2)) is given by:

$$\mathbf{y}(x) = \exp\left[\int_{x_\mathbf{I}}^{x}[\mathbf{A_D}(x)]dx\right]\left[\mathbf{y}(x_\mathbf{I}) + \int_{x_I}^{x}\mathbf{b_D}(x)\exp\left[-\int_{x_\mathbf{I}}^{x}[\mathbf{A_D}(x)]dx\right]dx\right] = [\mathbf{A}(x_I,x)]\mathbf{y}(x_\mathbf{I}) + \mathbf{b}(x_I,x), \tag{3}$$

where,

$[\mathbf{A}(x_I,x)] = \exp[\int_{x_\mathbf{I}}^{x}[\mathbf{A_D}(x)]dx]$ is the transfer matrix from a general point x to the initial x_I.

$\mathbf{b}(x_I,x) = \exp\left[\int_{x_\mathbf{I}}^{x}[\mathbf{A_D}(x)]dx\right]\int_{x_I}^{x}\mathbf{b_D}(x)\exp\left[-\int_{x_\mathbf{I}}^{x}[\mathbf{A_D}(x)]dx\right]dx$ is the vector transmitted from initial x_I to a general point x.

Previous solution particularized for both points x_I and x_{II}, gives the next relation:

$$\mathbf{y}(x_\mathbf{II}) = \exp\left[\int_{x_\mathbf{I}}^{x_\mathbf{II}}[\mathbf{A_D}(x)]dx\right][\mathbf{y}(x_\mathbf{I}) + \int_{x_\mathbf{I}}^{x_\mathbf{II}}\mathbf{b_D}(x)\exp\left[-\int_{x_\mathbf{I}}^{x}[\mathbf{A_D}(x)]dx\right]dx] = [\mathbf{A}(x_\mathbf{I},x_\mathbf{II})]\mathbf{y}(x_\mathbf{I}) + \mathbf{b}(x_\mathbf{I},x_\mathbf{II}), \tag{4}$$

where,

$[\mathbf{A}(x_\mathbf{I},x_\mathbf{II})] = \exp\left[\int_{x_\mathbf{I}}^{x_\mathbf{II}}[\mathbf{A_D}(x)]dx\right]$ is the *exact analytical transfer matrix.*

$\mathbf{b}(x_\mathbf{I},x_\mathbf{II}) = \exp\left[\int_{x_\mathbf{I}}^{x_\mathbf{II}}[\mathbf{A_D}(x)]dx\right]\int_{t_\mathbf{I}}^{t_\mathbf{II}}\mathbf{b_D}(x)\exp\left[-\int_{x_\mathbf{I}}^{x}[\mathbf{A_D}(x)]dx\right]dx$ is the *vector transferred.*

2.2. Numerical solution by finite transfer method of first order

Applying the first order approximation to the differential system (Eq. (2)) gives [7]:

$$\frac{d\mathbf{y}(x)}{dx} \cong \frac{\Delta\tilde{\mathbf{y}}(x_i)}{\Delta x} = \frac{\tilde{\mathbf{y}}(x_{i+1}) - \tilde{\mathbf{y}}(x_i)}{\Delta x} = [\mathbf{A_D}(x_i)]\tilde{\mathbf{y}}(x_i) + \mathbf{b_D}(x_i). \tag{5}$$

Thus, the finite transfer equation in this case is:

$$\tilde{\mathbf{y}}(x_{i+1}) = [[I] + [\mathbf{A_D}(x_i)]\Delta x]\tilde{\mathbf{y}}(x_i) + \mathbf{b_D}(x_i)\Delta x. \tag{6}$$

If we use the above relation, we can write the expression of the functions at a general point x_{i+1} in terms of the initial point x_I using a recurrence scheme:

$$\tilde{\mathbf{y}}(x_{i+1}) = \prod_{j=0}^{j=i}[[\mathbf{I}] + [\mathbf{A_D}(x_j)]\Delta x]\tilde{\mathbf{y}}(x_\mathbf{I}) + \sum_{j=0}^{j=i}\left[\prod_{k=j+1}^{k=i}[[\mathbf{I}] + [\mathbf{A_D}(x_k)]\Delta x]\right]\mathbf{b_D}(x_j)\Delta x = [\mathbf{A}(x_\mathbf{I},x_{i+1})]\tilde{\mathbf{y}}(x_\mathbf{I}) + \mathbf{b}(x_\mathbf{I},x_{i+1}). \tag{7}$$