Applied Mathematical Modelling 38 (2014) 5922–5933

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Solving the random diffusion model in an infinite medium: A mean square approach

M.-C. Casabán *, R. Company, J.-C. Cortés, L. Jódar

Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, Building 8G Access C 2nd floor, Camino de Vera s/n, 46022 Valencia, Spain

ARTICLE INFO

Article history:
Received 3 May 2013
Received in revised form 29 December 2013
Accepted 29 April 2014
Available online 15 May 2014

Keywords:
Random diffusion
Mean square approach
Infinite medium
Analytic-numerical solution
Random Fourier transform

ABSTRACT

This paper deals with the construction of an analytic-numerical mean square solution of the random diffusion model in an infinite medium. The well-known Fourier transform method, which is used to solve this problem in the deterministic case, is extended to the random framework. Mean square operational rules to the Fourier transform of a stochastic process are developed and stated. The main statistical moments of the stochastic process solution are also computed. Finally, some illustrative numerical examples are included.

1. Introduction

Diffusion models involve uncertainties not only to the error measurements but also due to material defects or impurities in the case of heat diffusion [1–3]. Uncertainty also appears when in a diffusion model one considers pollutants which present impurities.

The theory and applications of random differential equations is a very active area of mathematical research and there are several different approaches, from the so-called stochastic differential approach based on the Itô calculus, to the called dishonest methods [4]. Fruitful methods approach have been recently used in [5–10]. Random heat transfer in a finite medium have been treated in [2] by developing random perturbation method, by finite elements method in [1], and finite difference method in [11,12]. A different random approach focussing on Brownian motion stochastic processes have also been treated in [13,14] for the finite medium model and it is based on the Itô calculus. Here we follow the mean square approach developed for both the ordinary and partial differential case in [15–19]. This approach has two suitable properties. The first is that our solution coincides with the one of the deterministic case, i.e., when the random data are deterministic. The second, is that if $X_n(t)$ is a mean square approximation to the exact mean square solution $X(t)$, then the expectation and the variance of $X_n(t)$ converges to the expectation and the variance of $X(t)$, respectively [20].

For the sake of clarity in the presentation of the paper and thinking of applications, we assume that randomness is expressed by modeling the diffusion coefficient as a random variable (r.v.) and the initial stochastic process (s.p.) as a function that depends on a random variable. The same results are available, but with a more complicated notation by considering

* Corresponding author. Tel.: +34 (96)3879144; fax: +34 963877669.
E-mail addresses: macabar@imm.upv.es (M.-C. Casabán), rcompany@imm.upv.es (R. Company), jccortes@imm.upv.es (J.-C. Cortés), ljodar@imm.upv.es (L. Jódar).

http://dx.doi.org/10.1016/j.apm.2014.04.063

functions which depend on a finite number of random variables, the so-called functions with a finite degree of randomness quoted in [20, p. 37]. In this paper, we consider the one-dimensional random diffusion model

$$u_t = Au_{xx}, \quad -\infty < x < +\infty, \quad 0 < t < +\infty, \tag{1}$$

$$u(x,0) = \varphi(x;B), \quad -\infty < x < +\infty, \tag{2}$$

where A is a positive r.v., independent of r.v. B and both satisfying additional properties to be specified later. We will also specify later the further condition to be satisfied by the s.p. $\varphi(x;B)$. Unlike to the finite medium random diffusion model, to the best of our knowledge there is a lack of reliable numerical answers to the solution of the random diffusion model in an infinite medium. This paper deals with the construction of reliable solutions to model (1) and (2) by extending to the random case the Fourier transform approach, but focussing more on the applications than in theoretical issues. It is important to point out that in the random case we are not only interested in the construction of a solution s.p. $u(x,t)$ but we also need to compute the expectation and the variance of $u(x,t)$.

This paper is organized as follows. Section 2 is devoted to some preliminaries that will clarify both the understanding and reading of the paper. Section 3 addresses the definition of mean square Fourier transform, some relevant properties and a useful example and operational rules. In Section 4, the solution of problem (1) and (2) is performed as well as its statistical moments. In Section 5 some illustrative examples are included and conclusions are drawn in Section 6.

2. Preliminaries about random mean square calculus

This section begins by reviewing some important concepts, definitions and results related to the random L_p calculus, mainly focusing on the mean square (m.s.) and mean fourth (m.f.) calculus, which correspond to $p = 2$ and $p = 4$, respectively (see [17] for further details). After a relevant class of r.v.'s that will play an important role in the development of next sections is studied.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probabilistic space. Let $p \geqslant 1$ be a real number. A real r.v. U defined on $(\Omega, \mathcal{F}, \mathcal{P})$ is called of order p, if

$$\mathrm{E}[|U|^p] < +\infty,$$

where $\mathrm{E}[\cdot]$ denotes the expectation operator. The space L_p of all the real r.v.'s of order p, endowed with the norm

$$\|U\|_p = (\mathrm{E}[|U|^p])^{1/p} \tag{3}$$

is a Banach space, [21, p. 9].

Let $\{U_n : n \geqslant 0\}$ be a sequence of r.v.'s of order p. We say that it is convergent in the pth mean to the real r.v. $U \in L_p$, if

$$\lim_{n\to+\infty} \|U_n - U\|_p = 0.$$

If $p_2 \geqslant p_1$, then $L_{p_2} \subseteq L_{p_1}$. In addition if $\{U_n : n \geqslant 0\}$ is p_2th mean convergent to $U \in L_{p_2}$, then $\{U_n : n \geqslant 0\}$ is also p_1th mean convergent to $U \in L_{p_1}$, [21, p. 13]. Convergences in L_2 and L_4 are usually referred to as m.s. and m.f. convergence, respectively. If U and V are r.v.'s in L_4 then by the Schwarz's inequality one gets (see [17])

$$\|UV\|_2 \leqslant \|U\|_4 \|V\|_4. \tag{4}$$

If $\{U_n : n \geqslant 0\}$ is a sequence of 2-r.v.'s in L_2 m.s. convergent to $U \in L_2$, then from Theorem 4.3.1 of [20, p. 88] one gets

$$\lim_{n\to\infty} \mathrm{E}[U_n] = \mathrm{E}[U], \quad \lim_{n\to\infty} \mathrm{Var}[U_n] = \mathrm{Var}[U], \tag{5}$$

where $\mathrm{Var}[\cdot]$ denotes the variance operator. Let T be a subset of the real line. A family $\{U(t) : t \in T\}$ of real r.v.'s of order p is said to be a s.p. of order p or, in short, a p-s.p. if

$$\mathrm{E}[|U(t)|^p] < +\infty, \quad \forall t \in T.$$

We say $\{U(t) : t \in T\}$ is pth mean continuous at $t \in T$, if

$$\|U(t+h) - U(t)\|_p \to 0 \quad \text{as} \quad h \to 0, \quad t, \quad t+h \in T.$$

Furthermore, if there exists a s.p. $U'(t)$ of order p, such that

$$\left\| \frac{U(t+h) - U(t)}{h} - U'(t) \right\|_p \to 0 \quad \text{as} \quad h \to 0, \quad t, \quad t+h \in T,$$

then we say that $\{U(t) : t \in T\}$ is p th mean differentiable at $t \in T$ and $U'(t)$ is the p-derivative of $U(t)$.

In the particular cases that $p = 2,\ 4$, definitions above leads to the corresponding concepts of mean square (m.s.) and mean fourth (m.f.) continuity and differentiability. Furthermore, it is easy to prove by (4) that m.f. continuity and differentiability entail m.s. continuity and differentiability, respectively.

In accordance with [20, p. 99], [22], we say that a s.p. $\{V(x) : x \in \mathbb{R}\}$ with $V(x) \in L_p$ for all x, is locally integrable in $\mathbb{R}$ if, for all finite interval $[a,b] \subset \mathbb{R}$, the integral