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Short communication

A new exact solution of a damped quadratic non-linear oscillator

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ABSTRACT

In this paper, we derive a new exact solution of the damped quadratic nonlinear oscillator (Helmholtz oscillator) based on the developed solution for the undamped case by the Jacobi elliptic functions. It is interesting to see that both of the damped Duffing oscillator and Helmholtz oscillator possess solutions that follow closely to the undamped case, and even the solution procedures are almost the same.

1. Introduction

This paper deals with the derivation of exact solution of the damped quadratic nonlinear oscillator

$$\ddot{x} + 2\nu\dot{x} + \alpha x + \varepsilon x^2 = 0, \tag{1}$$

where x denotes the displacement of the system, ε is a non-linear system parameter, ν is the damping factor, and α can be seen as the linear stiffness. The dynamics of oscillator of this kind mimics the dynamics of certain pre-stressed structures, the capsizing of a ship [1] and nonlinear dynamics of a drop in a time-periodic flow [2] or in a time-periodic electric field [3]. It appears also in relation to the randomization of solitary-like waves in boundary-layer flows [4] and in the three-wave interaction, also referred to as a resonant triad [5].

This system has received some attention by different authors. Using the Lie theory for differential equation, Almendral and Sanjuán [6] found a parametric choice $\alpha = 24\nu^2/25$, for which the system is integrable and derived a class of exact solutions in terms of hyperbolic and Jacobian elliptic functions for this parametric value. Notice that they made an assumption that $\nu,\ \alpha$ are positive constants while ε is negative. The same result was also obtained by Chandrasekar et al. [7] through the modified Prelle–Singer method without making any assumption to parameters ν, α and ε. Feng and Meng [8] claimed that Chandrasekar et al. missed another class of exact solutions which could be derived by the Lie symmetry method when $\alpha = -24\nu^2/25$, which is a good complementary to [6]'s results. Note that this case is not that of an oscillator because at small x, the 'restoring force' is really repulsive.

On the other hand, it is known that the damped Duffing equation

$$\ddot{x} + 2\nu\dot{x} + \alpha x + \varepsilon x^3 = 0 \tag{2}$$

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possesses a solution that follows closely the undamped case [9–11] i.e. Eq. (2) has an exact solution of the form

$$x(t) = a(t)\text{cn}[\omega(t), m] \tag{3}$$

in which parameters a, ω, and m can be determined by substituting Eq. (3) to Eq. (2). And recently, Elías-Zúñiga [12] derived a highly accurate analytical solution to the damped Helmholtz–Duffing equation

$$\ddot{x} + 2v\dot{x} + \alpha x + Bx^2 + \varepsilon x^3 = 0, \tag{4}$$

which has a rational form elliptic function

$$x(t) = \frac{a(t) - b(t) + c(t)[a(t) + b(t)]\text{cn}[\omega(t), m]}{1 + c(t)\text{cn}[\omega(t), m]}, \tag{5}$$

where parameters a, b, c, ω, and m can be determined by substituting Eq. (5) to Eq. (4). Notice that (5) with constant parameters a, b, c, m, and $\omega(t) = \omega t$ is the exact solution of Eq. (4) when $v = 0$ [13].

Motivated by the derivation of solutions (3) and (5), it is expected that Eq. (1) possesses a general solution as

$$x(t) = a_1(t) + a_2(t)\text{sn}^2[\omega(t), m]. \tag{6}$$

Notice that (6) with constant parameters a_1, a_2 and $\omega(t) = \omega t$ is the exact solution of Eq. (1) when $v = 0$ [14]. If this conjecture is correct, then it will become a very interesting phenomenon in mathematics, and the solution will be much more concise than these in paper [6–8].

The organization of the paper is as follows. In Section 2, the solving process of Eq. (2) will be reviewed, and then in Section 3, the exact solution of Eq. (1) will be derived by almost the same method in Section 2. Also the relationship between this solution and the already existed solutions in [6] or [8] will be discussed. In Section 4 the exact solution will be compared with the numerical simulink to verify the correctness of the derived solution. Finally, Section 5 provides the concluding remarks.

2. Solving process of the damped Duffing oscillator

For solving the damped Duffing equation, we assume that the exact solution of Eq. (2) is as the form of Eq. (3)

$$x(t) = a(t)\text{cn}[\omega(t), m], \tag{3}$$

where $\text{cn}[\omega(t), m]$ is the cn Jacobian elliptic function, a, ω, and m are parameters to be determined. Substituting (3) into Eq. (2) yields

$$\begin{aligned}&\text{cn}[\omega(t), m][\alpha a(t) + 2va'(t) - a(t)\omega'(t)^2 + 2ma(t)\omega'(t)^2 + a''(t)] + \text{cn}^3[\omega(t), m][\varepsilon a(t)^3 - 2ma(t)\omega'(t)^2]\\ &\quad - \text{sn}[\omega(t), m]\text{dn}[\omega(t), m][2va(t)\omega'(t) + 2a'(t)\omega'(t) + a(t)\omega''(t)] = 0,\end{aligned} \tag{7}$$

where $\text{sn}[\omega(t), m]$ and $\text{dn}[\omega(t), m]$ are the sn and dn Jacobian elliptic function, and the following identities for cn, sn and dn has been used

$$\text{sn}^2 + \text{cn}^2 = 1; \quad \text{dn}^2 + m\,\text{sn}^2 = 1. \tag{8}$$

Note that Eq. (7) holds for all time t if and only if

$$\alpha a + 2va' - a\omega'^2 + 2ma\omega'^2 + a'' = 0, \tag{9}$$

$$\varepsilon a^3 - 2ma\omega'^2 = 0, \tag{10}$$

$$2va\omega' + 2a'\omega' + a\omega'' = 0. \tag{11}$$

Thus, we get 3 equations for 3 unknown parameters. Notice that a cannot be zero, so we can get the following equation from (10)

$$a^2 = \frac{2m}{\varepsilon}\omega'^2. \tag{12}$$

Taking derivative of (12) gives

$$aa' = \frac{2m}{\varepsilon}\omega'\omega''. \tag{13}$$

We have two methods to solve Eqs. (9)–(11).

2.1. The first method

Multiplying (11) by a and using (12) and (13) yields