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# Modeling the packing coloring problem of graphs

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ABSTRACT

The frequency assignment problem asks for assigning frequencies to transmitters in a wireless network and includes a variety of specific subproblems. In particular, the packing coloring problem comes from the regulations concerning the assignment of broadcast frequencies to radio stations. The problem is placed on a graph-theoretical footing and modeled as the packing chromatic number $\chi_\rho(G)$ of a graph $G$. The packing chromatic number of $G$ is the smallest integer $k$ such that the vertex set $V(G)$ can be partitioned into disjoint classes $X_1, \dots, X_k$, with the condition that vertices in $X_i$ have pairwise distance greater than $i$. The determination of the packing chromatic number is known to be **NP**-hard. This paper presents an integer linear programming model and a satisfiability test model for the packing coloring problem. The proposed models were applied for studying the packing chromatic numbers of Cartesian products of paths and cycles and for the packing chromatic numbers of some distance graphs.



## 1. Introduction

Wireless communication is applied in various situations such as mobile telephony, radio and TV broadcasting, satellite communication, and military operations. In each of these applications a frequency assignment problem arises with application specific characteristics. Hence, different modeling ideas for each of the features of the problem, such as the treating of interference among wireless signals, the availability of frequencies, and the optimization criterion have been developed [1].

The *frequency assignment problem* asks for assigning frequencies to transmitters in a wireless network. In a broadcasting network, each transmitter is assigned a frequency channel for its transmissions. The spectrum of frequencies gets more and more scarce, because of increasing demands, both civil and military. Thus the task is to minimize the span of frequencies while avoiding interference.

The frequency assignment problem has inspired a variety of graphical coloring problems. One of its well elaborated graph theoretical models is the concept of *distance constrained labeling of graphs* [2]. Here it is assumed that the distance of transmitters can be modeled by a graph, and that the distance of the transmitters influences possible interference in such a way that the closer two transmitters are, the farther apart their frequencies must be.

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Not only because of their interesting theoretical properties but also for their practical applications, some other graphical coloring concepts inspired by the frequency assignment problem have received a lot of attention in recent literature. The approach that we follow in the present paper is the *the packing coloring problem* which comes from the regulations concerning the assignment of broadcast frequencies to radio stations. In particular, two radio stations which are assigned the same broadcast frequency must be located sufficiently far apart so that neither broadcast interferes with the reception of the other. Two radio stations which are assigned the same frequency must be placed sufficiently far apart so that neither broadcast interferes with the reception of the other. Moreover, the geographical distance between two radio stations which are assigned the same frequency is directly related to the power of their broadcast signals.

Formally, a $k$-coloring of a graph $G$ is a function $f$ from $V(G)$ onto a set $C = \{1, 2, \ldots, k\}$ (with no additional constraints). The elements of $C$ are called *colors*. Let $X_i$ denote the set of vertices with the image (color) $i$. Note that $X_1, \ldots, X_k$ is partition of the vertex set of $G$ into disjoint (color) classes.

Let $X_1, \ldots, X_k$ be a partition of the vertex set of $G$ with respect to the following constraints: each color class $X_i$ is a set of vertices with the property that any distinct pair $u,\ v \in X_i$ satisfies $d_G(u, v) > i$. Here $d_G(u, v)$ denotes the usual shortest path distance between $u$ and $v$. Then $X_i$ is said to be an *i-packing*, while such a partition is called a *packing k-coloring*. The smallest integer $k$ for which there exists a packing $k$-coloring of $G$ is called the *packing chromatic number of G* and it is denoted by $\chi_\rho(G)$.

As noticed in [3], the concept could have several additional applications, as for instance in resource placements and biological diversity. In particular, different species in a certain area require different amount of territory.

The determination of the packing chromatic number is known to be **NP**-hard for general graphs [4]. In addition, it is also proved [4] that it is **NP**-complete to decide whether $\chi_\rho(G) \leqslant 4$. But things are even worse: Fiala and Golovach [5] showed that determining $\chi_\rho(G)$ is one of few problems that are **NP**-complete on trees.

Product graphs are considered in order to gain global information from the factor graphs [6]. Many interesting interconnection networks are based on Cartesian product graphs with simple factors, such as paths and cycles. In particular, any square lattice is the Cartesian product of two paths and any torus is the Cartesian product of two cycles. Most of the work on the frequency assignment problem has dealt with grids and torus by studying the distance constrained labeling of graphs, e.g. see [7] and the references therein.

The research of the packing chromatic number started by investigating the packing chromatic number of the square lattice. Goddard et al. [4] determined the packing chromatic number for infinite subgraphs of the square lattice with up to 5 rows. In the same paper, the question of determining the packing chromatic number of the infinite square lattice was posed and bounds on the packing chromatic number of the square lattice were determined. The upper bound was improved to 17 by Holub and Soukal [8], while Ekstein et al. [9] showed that a packing 11-coloring of the square lattice cannot exist. Some finite and infinite subgraphs of the square lattice as well as some related graphs, in particular the Cartesian products of paths and cycles, have been also extensively explored [3,9–11,8,12].

The study of a packing coloring of distance graphs $D(k, t)$ was initiated by Togni [13] who determined the packing chromatic number of $D(1, t)$ for small values of $t$. Distance graphs were further investigated by Ekstein et al. in [14,15], where some more results on the packing chromatic number of $D(1, t)$ and $D(k, t)$ are given.

This main contribution of the paper is an integer linear programming model and a satisfiability test model for the packing coloring problem of graphs presented in Section 2. The proposed models outperform other exact methods such as a backtracking and dynamic algorithm [12]. In particular, the packing chromatic numbers and improved bounds have been found for the Cartesian products of paths and cycles as presented in Section 3. We conclude the paper with the results on the packing chromatic number for some families of distance graphs provided in Section 4.

## 2. Modeling the packing coloring problem

### 2.1. Integer linear programming formulation

We consider a given graph $G = (V, E)$. For a vertex $v \in V(G)$ and an integer $i \in \{1, 2, \ldots, k\}$, let $x_{v,i}$ equal 1 if $v$ is labeled with $i$, 0 otherwise. The problem of finding the packing chromatic number of a graph $G$ can be formulated as an integer linear programming as follows:

**ILP 1:**

$$\text{minimize } z \tag{1}$$

subject to:

$$\sum_{i=1}^{k} x_{v,i} = 1, \quad \forall v \in V(G); \tag{2}$$

$$x_{v,i} + x_{u,i} \leqslant 1, \qquad d(v,u) \leqslant i, \quad \forall v,u \in V(G),\ 1 \leqslant i \leqslant k; \tag{3}$$

$$i x_{v,i} \leqslant z, \qquad \forall v \in V(G),\ 1 \leqslant i \leqslant k. \tag{4}$$

$$x_{v,i} \in \{0,1\}, \quad v \in V(G),\ 1 \leqslant i \leqslant k; \tag{5}$$