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A parameter estimation method based on random slow manifolds ☆

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ABSTRACT

A parameter estimation method is devised for a slow–fast stochastic dynamical system, where only the slow component is observable. By using available observations on the slow component, a system parameter is estimated by studying the slow system on the random slow manifold. This offers a benefit of dimension reduction in quantifying parameters in stochastic dynamical systems. An example is presented to illustrate this method, and to verify that the parameter estimator based on the lower dimensional, reduced slow system is a good approximation of the parameter estimator for the original slow–fast stochastic dynamical system.

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1. Introduction

Invariant manifolds provide geometric structures for understanding dynamical behavior of nonlinear systems under uncertainty. Some systems evolve on fast and slow time scales, and may be modeled by coupled singularly perturbed stochastic ordinary differential equations (SDEs). A slow–fast stochastic system may have a special invariant manifold called a random slow manifold that captures the slow dynamics.

We consider a stochastic slow–fast system

$$\dot{x} = Ax + f(x,y), \quad x(0) = x_0 \in \mathbb{R}^n, \tag{1.1}$$

$$\dot{y} = \frac{1}{\varepsilon}By + \frac{1}{\varepsilon}g(x,y) + \frac{\sigma}{\sqrt{\varepsilon}}\dot{W}_t, \quad y(0) = y_0 \in \mathbb{R}^m, \tag{1.2}$$

where A and B are matrices, ε is a small positive parameter measuring slow and fast scale separation, f and g are nonlinear Lipschitz continuous functions with Lipschitz constant L_f and L_g respectively, σ is a noise intensity constant, and $\{W_t : t \in \mathbb{R}\}$ is a two-sided $\mathbb{R}^m$-valued Wiener process (i.e., Brownian motion) on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Under a gap condition and

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the dissipative condition (**H1** and **H2** in Section 2) for matrix B, for ε sufficiently small, there exists a random slow manifold $(\xi, h^\varepsilon(\xi,\omega))$, with $h^\varepsilon(\xi,\omega) : \mathbb{R}^n \to \mathbb{R}^m, \omega \in \Omega$, as in [1,2], for slow–fast stochastic system (1.1) and (1.2). When the nonlinearities f, g are only locally Lipschitz continuous but the system has a random absorbing set (e.g., in mean-square norm), we conduct a cut-off of the original system.

The random slow manifold is the graph of a random nonlinear mapping $h^\varepsilon(\xi,\omega) = \sigma\eta^\varepsilon(\omega) + \tilde{h}^\varepsilon(\xi,\omega)$, with $\tilde{h}^\varepsilon(\xi,\omega)$ determined by a Lyapunov–Perron integral equation [1],

$$\tilde{h}^\varepsilon(\xi,\omega) = \frac{1}{\varepsilon}\int_{-\infty}^{0} e^{-\frac{B}{\varepsilon}s} g(x(s,\omega,\xi), y(s,\omega,\xi) + \sigma\eta^\varepsilon(\theta_s\omega))\, ds, \quad \xi \in \mathbb{R}^n,$$

here $\eta^\varepsilon(\omega) = \frac{1}{\sqrt{\varepsilon}}\int_{-\infty}^{0} e^{-\frac{B}{\varepsilon}s}\, dW_s$ and $\eta^\varepsilon(\theta_t\omega) = \frac{1}{\sqrt{\varepsilon}}\int_{-\infty}^{t} e^{\frac{B}{\varepsilon}(t-s)}\, dW_s$. The random slow manifold exponentially attracts other solution orbits. We will find an analytically approximated random slow manifold for sufficiently small ε, in terms of an asymptotic expansion in ε, as in [3,4]. This slow manifold may also be numerically computed as in [5]. Roberts [6] introduced a normal form transform method for stochastic differential systems with both slow modes and quickly decaying modes, in order to find the approximate formula for a slow manifold. Related works on the dynamics of stochastic differential equation or stochastic center manifold include [7–10]. By restricting to the slow manifold, we obtain a lower dimensional reduced system of the original slow–fast system (1.1) and (1.2), for ε sufficiently small

$$\dot{x} = Ax + f\left(x, \tilde{h}^\varepsilon(x, \theta_t\omega) + \sigma\eta(\theta_t\psi_\varepsilon\omega)\right), \quad x \in \mathbb{R}^n, \tag{1.3}$$

where θ_t and ψ_ε will be defined in the next section.

If the original slow–fast system (1.1) and (1.2) contains unknown system parameters, but only the slow component x is observable, we conduct parameter estimation using the slow system (1.3). Since the slow system is lower dimensional than the original system, this parameter estimator offers an advantage in computational cost, in addition to the benefit of using only observations on slow variables.

This paper is arranged as follows. In the next section, we obtain an approximated random slow manifold and thus the random slow system. Then in Section 3, we provide an error estimation for our parameter estimator, in terms of $\mathcal{O}(\varepsilon)$ (due to random slow reduction) and the observation error. Finally, we present a simple example in Section 4 to illustrate our method.

2. Random slow manifold and its approximation

In order to use the reduced system to estimate a parameter, we firstly give some results on slow manifold and its approximation [1,5,6,11], Roberts. The slow manifold is considered under a driving flow $(\Omega, \mathscr{F}, \mathbb{P}, \theta)$. Here $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space. And $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ is a flow on Ω which is defined as a mapping

$$\theta : \mathbb{R} \times \Omega \mapsto \Omega$$

satisfying

- $\theta_0 = id_\Omega$ (identity mapping on Ω),
- $\theta_s\theta_t = \theta_{s+t}$ for all $s, t \in \mathbb{R}$, and
- the mapping $(t,\omega) \mapsto \theta_t\omega$ is $(\mathfrak{B}(\mathbb{R}) \times \mathscr{F}, \mathscr{F})$–measurable and $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

By a random transformation

$$\begin{pmatrix} X \\ Y \end{pmatrix} := \mathscr{V}_\varepsilon(\omega, x, y) = \begin{pmatrix} x \\ y - \sigma\eta^\varepsilon(\omega) \end{pmatrix}, \tag{2.1}$$

we convert the SDE system (1.1) and (1.2) to the following system with random coefficients

$$\dot{X}(t) = AX(t) + f(X(t), Y(t) + \sigma\eta^\varepsilon(\theta_t\omega)), \tag{2.2}$$

$$\dot{Y}(t) = \frac{1}{\varepsilon}BY(t) + \frac{1}{\varepsilon}g(X(t), Y(t) + \sigma\eta^\varepsilon(\theta_t\omega)), \tag{2.3}$$

where $\eta^\varepsilon(\omega) = \frac{1}{\sqrt{\varepsilon}}\int_{-\infty}^{0} e^{-\frac{Bs}{\varepsilon}}\, dW_s$ is the stationary solution of linear system $dy^\varepsilon = \frac{B}{\varepsilon}y^\varepsilon dt + \frac{\sigma}{\sqrt{\varepsilon}}dW_t$. And $\theta_t : \Omega \to \Omega$ is the Wiener shift implicitly defined by $W_s(\theta_t\omega) = W_{t+s}(\omega) - W_t(\omega)$. Note that $\eta^\varepsilon(\theta_t\omega) = \frac{1}{\sqrt{\varepsilon}}\int_{-\infty}^{t} e^{\frac{B}{\varepsilon}(t-s)}\, dW_s$.

Define a mapping (between random samples) $\psi_\varepsilon : \Omega \to \Omega$ implicitly by $W_t(\psi_\varepsilon\omega) = \frac{1}{\sqrt{\varepsilon}}W_{t\varepsilon}(\omega)$. Then $\frac{1}{\sqrt{\varepsilon}}W_{t\varepsilon}(\omega)$ is also a Wiener process with the same distribution as $W_t(\omega)$. Moreover, $\eta^\varepsilon(\theta_{t\varepsilon}\omega)$ and $\eta^\varepsilon(\omega)$ are identically distributed with $\eta(\theta_t\psi_\varepsilon\omega) = \int_{-\infty}^{t} e^{B(t-s)}\, dW_s(\psi_\varepsilon\omega)$ and $\eta(\psi_\varepsilon\omega) = \int_{-\infty}^{0} e^{-Bs}\, dW_s(\psi_\varepsilon\omega)$, respectively.

By a time change $\tau = t/\varepsilon$ and using the fact that $\eta^\varepsilon(\theta_{\tau\varepsilon}\omega)$ and $\eta(\theta_\tau\psi_\varepsilon\omega)$ are identically distributed, the system (2.2) and (2.3) is reformulated as