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Derivation, interpretation, and analog modelling of fractional variable order derivative definition

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ABSTRACT

The paper presents derivation and interpretation of one type of variable order derivative definitions. For mathematical modelling of considering definition the switching and numerical scheme is given. The paper also introduces a numerical scheme for a variable order derivatives based on matrix approach. Using this approach, the identity of the switching scheme and considered definition is derived. The switching scheme can be used as an interpretation of this type of definition. Paper presents also numerical examples for introduced methods. Finally, the idea and results of analog (electrical) realization of the switching fractional order integrator (of orders 0.5 and 1) are presented and compared with numerical approach.

1. Introduction

In this paper a variable fractional order derivative will be investigated. The fractional order derivatives and integrals are operators defined in the fractional calculus. This calculus is an extension of classical integer order differential calculus, for the case when orders of integration and differentiation actions are real or even complex. Idea of this generalization is nearly as old as the differential calculus itself and first time has been mentioned in correspondence between Leibniz and L'Hospital in 1695. From the fractional calculus point of view the traditional differential calculus is only special case when orders are integer. Thus, all analysis presented in this paper will be also valid for traditional integer order derivatives and integrals.

Theoretical formulation of the fractional calculus can be found in [1–3]; applications of this calculus to control are presented in [4], and to signal processing in [5]. Some results concerning fractional differential equations were presented in [6–8]. It can be efficiently used for modelling behavior or dynamics of many materials and systems, especially the diffusion based processes. For example, ultracapacitors are electrical devices that can be modeled more efficiently by fractional calculus. Due to ions diffusion in electrolyte and porous media, dynamics of such devices are different than traditional capacitors and were successfully modelled with fractional calculus based models in [9–12].

When the properties of a system or phenomena are changing in time, not only because of varying parameters but also because of structure changing, i.e., due to chemical reactions and temperature processes, variable order derivative and integral operators should be used for efficient modelling. The variable order behavior of one electrochemical device, confirmed by experimental data, was presented in [13]. History of drag expression was successfully modelled using variable order equations in [14]. Moreover, the variable order operators can be also applied to describe variable order fractional noise [15,16], and also to obtain new control algorithms [17]. Some results concerning stability of variable order systems are

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presented in [18] and its system properties in [19,20]. In [21], the variable order interpretation of the analog realization of fractional orders integrators, was presented. The applications of variable order derivatives and integrals can be found also in signal processing [5]. Some numerical methods for realization of variable order operators are presented in [13,22,23].

Description of variable order systems is much more complicated than for constant order case, mostly due to variety of variable order operator definitions [24,25]. Additionally, there was a lack of clear connection between the mathematical formulas, given by definitions, and interpretation of the changing order process. In this paper we present an identity between one type of definition and one type of switching strategy. This can be found as an interpretation of how the order is changing in time and can be very useful tool for further analysis.

The rest of the paper is organized as follows. Section 2 presents existing generalizations of Grunwald–Letnikov definition of fractional order derivatives. In Section 3, a switching scheme for practical implementation of variable order derivative is given and studied. Section 3 presents also a generalization of the matrix approach for switching order and derivation of identity of the switching scheme and the second type of definition. Section 4 presents numerical examples of the proposed methods compared to the analytical solutions. Finally, Section 5 presents an analog realization of the switched order integrator and comparison of obtained results to the numerical solutions.

2. Fractional variable order Grunwald–Letnikov type derivatives

As a base of generalization onto variable order derivative the following definition is taken into consideration:

Definition 1. Fractional constant order derivative is defined as follows:

$$ {}_0\mathrm{D}_b^{\alpha} f(t) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{r=0}^{n}(-1)^r\binom{\alpha}{r}f(b-rh), $$

where $h > 0$ is a step time, and $n = \lfloor b/h \rfloor$.

According to this definition, one obtains: fractional order derivatives for $\alpha > 0$, fractional order integrals for $\alpha < 0$, and the original function $f(t)$ for $\alpha = 0$. For the case of order changing with time (variable order case), three types of definition can be found in the literature [24,25]. The first one is obtained by replacing of constant order α by variable order $\alpha(t)$. In that approach, all coefficients for past samples are obtained for present value of the order and is given as follows:

Definition 2. The 1st type of fractional variable order derivative is defined as follows:

$$ {}_0\mathrm{D}_b^{\alpha(t)} f(t) = \lim_{h\to 0}\frac{1}{h^{\alpha(b)}}\sum_{r=0}^{n}(-1)^r\binom{\alpha(b)}{r}f(b-rh). $$

In Fig. 1, plots of unit step function $1(t)$ derivatives (according to Definition 2) are presented for $\alpha_1(t) = -1, \alpha_2(t) = -2$, and

$$ \alpha_3(t) = \begin{cases} -1 & \text{for } 0 \leqslant t < 1, \\ -2 & \text{for } 1 \leqslant t \leqslant 2. \end{cases} \tag{1} $$

The second type of definition assumes that coefficients for past samples are obtained for order that was present for these samples. In this case, the definition has the following form:

Definition 3. The 2nd type of fractional variable order derivative is defined as follows:

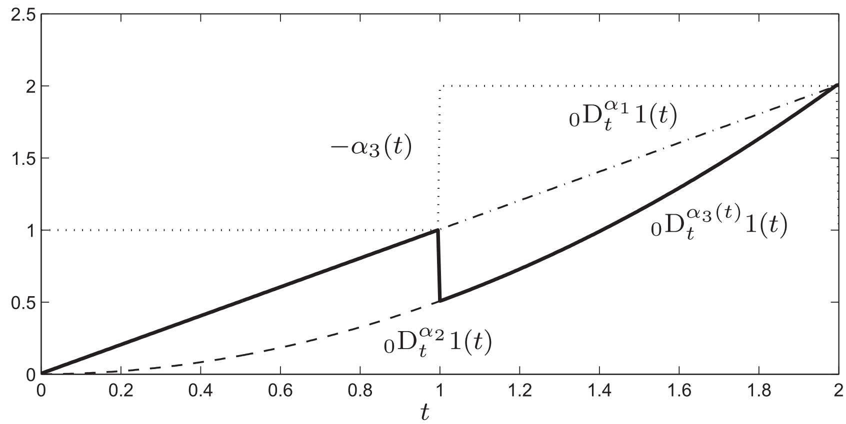

Fig. 1. Plots of unit step function derivatives with respect to the 1st type derivative (given by Definition 2).