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# A 3-dimensional well model in the flow transport through porous media

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ABSTRACT

In petroleum extraction and exploitation, the well is usually treated as a point or line source, due to its radius is much smaller comparing with the scale of the whole reservoir. Especially, in 3-dimensional situation, the well is regarded as a line source. In this paper, we analyze the modeling error for this treatment for steady flows through porous media and present a new algorithm for line-style well to characterize the wellbore flow potential. We also provide a numerical example to demonstrate the effectiveness of the proposed method.



## 1. Introduction

In many practical applications, especially by resistivity well-logging in petroleum exploitation, the boundary value problem with equivalued surfaces is formulated, see [1–4] for instance. From the physical view, the equivalued surface boundary corresponds to a source in the reservoirs. In 2-dimensional case, the equivalued surface boundary can be regarded as a point. So this type of source reduces to a point source, and the modeling error has been discussed by Chen–Yue [5]. In 3-dimensional situation, the equivalued surface boundary can be treated as a line rather than a point. One main aim of the present paper is to derive the modeling error in 3-dimensional case.

Consider a vertical well in $\mathbb{R}^3$. Let $\Omega = \Theta \times (a,d) \subset \mathbb{R}^3$ be a cylinder-like domain, where $\Theta \subset \mathbb{R}^2$ is a bounded domain. For a point $(x_0, y_0) \in \Theta$, denote by $B_\delta$ the small disk contained in $\Theta$ with center $(x_0, y_0)$ and radius $\delta$. Let $B_\delta \times (a,d)$ be the domain that occupied by the well. Set $\Theta_\delta = \Theta \setminus \overline{B}_\delta$ and $\Omega_\delta = \Theta_\delta \times (a,d)$. We consider the following single phase potential equation which is formed by combining Darcy's Law with conversation of mass

$$-\mathrm{div}(K\nabla u_\delta) = 0 \quad \text{in } \Omega_\delta,$$

where $u_\delta$ is the flow potential, $K$ is the permeability. We will impose mixed boundary conditions

$$u_\delta|_{\partial\Theta\times(a,d)} = 0, \quad \left.\frac{\partial u_\delta}{\partial \nu}\right|_{\Theta_\delta\times\{a,d\}} = 0,$$

on the exterior boundary. On the well boundary $\partial B_\delta \times (a,d)$, two quantities are of particular importance in practical applications: the wellbore potential $u|_{\partial B_\delta\times(a,d)}$ and the well flow rate $\int_{\partial B_\delta\times\{z\}} K\frac{\partial u_\delta}{\partial \nu}\,ds$ $(z \in (a,d))$, where $\nu$ is the unit outer

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normal to $\Omega_\delta$. In practice, the well is only open in part segment. Let $[a,d]=[a,b]\cup(b,c)\cup[c,d]$. Suppose that the well is open in $(b,c)$ and closed in $[a,b]\cup[c,d]$. Therefore we set up two kinds of boundary conditions for the wellbore boundary. Namely we assume that the layer flow rate is fixed in $(b,c)$ and no flow rate in $[a,b]\cup[c,d]$, i.e.

$$u_\delta|_{\partial B_\delta\times\{z\}}=\mu(z)\ (\text{unknown}),\quad \int_{\partial B_\delta\times\{z\}} K\frac{\partial u_\delta}{\partial \nu}ds=q(z),\quad z\in(a,d),$$

where $\mu(\cdot)$ is a unknown function which depends only the height $z$ of the well and $q(z)$ is defined by

$$q(z)=\begin{cases} q(z) & z\in(b,c),\\ 0 & z\in[a,b]\cup[c,d].\end{cases} \tag{1.1}$$

In this paper, we mainly consider the following boundary value problem

$$\begin{cases} -\mathrm{div}(K(x,y,z)\nabla u_\delta)=0\ (x,y,z)\in\Omega_\delta,\\ u_\delta|_{\partial\Theta\times(a,d)}=0,\\ \frac{\partial u_\delta}{\partial\nu}|_{\Theta_\delta\times\{a,d\}}=0,\\ u_\delta|_{\partial B_\delta\times\{z\}}=\mu_0(z)\ (\text{unknown}),\quad \int_{\partial B_\delta\times\{z\}}K\frac{\partial u_\delta}{\partial\nu}ds=q(z),\quad z\in(a,d),\end{cases} \tag{1.2}$$

where $q(z)$ is defined in (1.1).

Since the size of the well $\delta$ is negligible in practical situations, the approximation of problem (1.2) is as follows

$$\begin{cases} -\mathrm{div}(K(x,y,z)\nabla u)=q(z)\delta_l\ (x,y,z)\in\Omega,\\ u|_{\partial\Theta\times[a,d]}=0,\\ \frac{\partial u}{\partial\nu}|_{\Theta\times\{a,d\}}=0.\end{cases} \tag{1.3}$$

Here $q(z)$ is the flow rate mentioned in (1.2), and $\delta_l$ is the line-style Dirac measure at $\{(x_0,y_0,z),z\in(b,c)\}$ defined by

$$\int_\Omega f(x,y,z)\delta_l d\Omega=\int_b^c f(x_0,y_0,z)dz,\quad \forall f\in C(\Omega).$$

The first result of this paper is the following error estimate between the exact solution $u_\delta$ and the approximate solution $u$, i.e.

$$\max_{x\in\bar\Omega_\delta}|u-u_\delta|\leqslant C\delta|\ln\delta|,$$

where the constant $C$ is independent of $\delta$. As far as we know, this is the first result in 3-dimensional case. The proof depends on a sharp estimate between the well flow rate and the wellbore potential.

Next we develop a new algorithm to get the wellbore potential on mesh $h\gg\delta$. The main idea of our algorithm is to indicate the wellbore flow by proposing a quantity named equivalent flow potential (EFP) $\frac{1}{b-c}\int_b^c u|_{\partial B_\delta}dz$ (where $u$ is the flow potential, $(b,c)$ is the height of well, and $\partial B_\delta\times(b,c)$ is the wellbore boundary). We first solve the problem (1.3) by finite element method and then obtain the discrete solution $u_h$. It is well known that $u_h$ fails to give good approximation in the vicinity of well singularity. One way to overcome this difficulty is to introduce a auxiliary function $w=u-\psi$, where $\psi=\frac{1}{4\pi K_0}\int_a^d\frac{q_0(l)}{\sqrt{(x-x_0)+(y-y_0)^2+(z-l)^2}}dl$ is the singular function.

Since $w$ is regular, we can do accurate approximation for $w$. Denote by $w_h\in V_h$ the finite element approximation of $w$. Thus the value of $u_h$ on the well boundary $\Gamma_\delta=\{(x,y,z):\sqrt{(x-x_0)^2+(y-y_0)^2}=\delta,z\in(c,d)\}$ can be approximated by

$$u_h|_{\Gamma_\delta}=w_h(x_0,y_0,z)+\psi|_{\partial B_\delta}.$$

Furthermore, $\int_b^c u|_{\partial B_\delta}dz$ can be approximated by

$$\bar\alpha_h=\int_b^c w_h(x_0,y_0,z)dz+\int_b^c\psi|_{\partial B_\delta}dz.$$

Our method improves the computational efficiency because we only use a quantity named EFP to characterize the wellbore pressure instead of correcting the wellbore potential at every layer.

The rest of the paper is organized as follows. In Section 2, we prove the error estimate between the exact solution and the approximate solution. In Section 3, we introduce our new algorithm and verify the convergence property of this algorithm. In Section 4, we give some numerical results to demonstrate the accuracy of our proposed method. We end the paper with some concluding remarks.

## 2. The modeling error

Let $D\subset\Omega$ be a subdomain with Lipschitz boundary. For each integer $m\geqslant0$ and $1\leqslant p\leqslant+\infty$, we denote by $W^{m,p}(D)$ the standard Sobolev space of real functions having all their weak derivatives up to order $m$ in the Lebesgue space $L^p(D)$. The