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Analytical solution for laterally loaded long piles based on Fourier–Laplace integral

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ABSTRACT

Piles are frequently used to support lateral loads. Elastic solutions based on the Winkler foundation model are widely used to design laterally loaded piles at working load. This paper reports a simplified analytical solution for laterally loaded long piles in a soil with stiffness linearly increasing with depth. Based on a Fourier–Laplace integral, a power series solution for small depth and a Wentzel–Kramers–Brillouin (WKB) asymptotic solution for large depth are derived. By using this analytical solution, the deflection and bending moment profiles of a laterally loaded pile can be obtained through simple calculation. The proposed power series solution is exact for infinitely long piles. Numerical examples show that this solution agrees well with other existing methods on predicting the deflection and bending moment of laterally loaded piles. The WKB asymptotic solution developed in this study has never been introduced before. The simplified analytical solution obtained in this study provides a better approach for engineers to analyze the responses and design of laterally loaded long piles.

1. Introduction

Piles are widely used to support laterally loaded structures, such as bridges, buildings, tanks, and wind turbines. Some analytical methods have been developed for analyzing such piles, including the elastic subgrade reaction approach by Matlock and Reese [1], Davisson and Gill [2], Shen and Teh [3] or the elastic continuum approach by Poulos and Davis [4], Zhang and Small [5], Shen and Teh [6]. Of these methods, the subgrade reaction approach based on Winkler foundation model is most widely used for its clear concept and simple mathematical treatment. Terzaghi [7] proposed that the modulus of subgrade reaction should be a constant with the depth of clay, whereas this modulus should increase linearly with depth from a value of zero at the ground surface for sand. By using the beam on elastic foundation model, Chang [8] derived an analytic solution for a laterally loaded pile in clay by assuming the coefficient of the subgrade reaction is a constant and the pile is sufficiently long. Several methods have been developed for the analysis of laterally loaded piles in sand, including the finite difference method by Gleser [9], Matlock and Reese [1], Reese and Matlock [10] and power series solution by Hetenyi [11], Rowe [12,13]. Finite difference solutions can be very close to the actual solution if sufficient segments are used.

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However, the efficiency of the calculation is relatively low especially when the pile is very long and a large number of segments are used. The power series expression is an approximate solution because the boundary conditions at the tip of pile can hardly be exactly satisfied by the finite terms of the power series. Moreover, more terms of the power series are needed with the increasing depth of piles to ensure accuracy, leading to a greater amount of calculation.

To analyze laterally loaded long piles with higher accuracy and simplicity, this paper proposes an analytical solution based on the Fourier–Laplace integral method, which recovers power series solutions for small depth and Wentzel–Kramers–Brillouin (WKB) asymptotic solutions for large depth. The power series solution is used to analyze the small depth part of the pile with high accuracy by use of only a few terms; while the WKB approximation is employed to analyze the large depth part of the pile with much less work but acceptable accuracy compared with the power series. For infinitely long piles, the proposed power series solution is an exact solution as the boundary conditions at the tip of the pile are satisfied exactly. Furthermore, the simplified analytical solutions to deflection and bending moment of laterally loaded long piles are obtained, which can be conveniently used by engineers to facilitate analysis and design of piles. In addition, the present method can also be extended to analyze laterally loaded long piles in soil with the modulus of subgrade reaction in some other functions of the depth. The method proposed in this article is also available to analyze a short pile which is addressed in Appendix B.

2. Definition of the problem

According to Winkler foundation model, the flexural equation of a pile on the elastic subgrade can be written as

$$E_pI_p\frac{d^4y}{dz^4}+K\cdot y=0, \tag{1}$$

where E_p is the Young's modulus of the pile, I_p is the inertia moment of the pile, y is the pile deflection, z is the pile depth, K is the modulus of subgrade reaction.

The modulus of subgrade reaction increases linearly with the depth from a value of zero at the ground surface for sand and can be written as after Poulos and Davis [4].

$$K=n_hz, \tag{2}$$

where n_h is the constant of horizontal subgrade reaction.

Substituting Eq. (2) into Eq. (1) yields

$$E_pI_p\frac{d^4y}{dz^4}+n_hzy=0. \tag{3}$$

3. Solution procedure

The relative stiffness factor, T, proposed by Reese and Matlock [10] is given by

$$T=\left(\frac{E_pI_p}{n_h}\right)^{\frac{1}{5}}. \tag{4}$$

By defining a dimensionless variable $x = z/T$, Eq. (3) can be reduced to

$$\frac{E_pI_p}{n_hT^5}\frac{d^4y}{dx^4}+xy=0. \tag{5}$$

Plugging Eq. (4) into Eq. (5) gives

$$\frac{d^4y}{dx^4}+xy=0. \tag{6}$$

Eq. (6) is the fundamental equation explored in this paper.

3.1. Fourier–Laplace integral representation for the solutions

In Shen and Teh [6], the solution procedure for the Airy Equation $y''(x)=xy$ is discussed using a Fourier–Laplace Integral representation. Here, we apply this representation to solve our equation.

Consider the Fourier–Laplace representation of $y(x)$ (after White [14]):

$$y(x)=\int_C e^{xt}f(t)dt, \tag{7}$$

where C is the contour in the complex plane with endpoints a and b.