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Two dimension spatial pattern formation in a coupled autocatalysis system ☆

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ABSTRACT

This paper addresses the cubic autocatalator kinetics modeling of coupling via diffusion interchange of autocatalyst. By incorporating the effect of two identical cells, each governed by cubic autocatalator kinetics, considering the possibility of the spatiotemporal structures of two dimensional Turing patterns, a new model is proposed. Unlike previous models, the proposed model has two dimensional spatial variation. First, the equations and the local stability are obtained by linearizing about the spatially uniform solutions. It is shown that the necessary condition for the model undergoes bifurcation by using the singular perturbation theory. Next Landau constant and amplitude functions of two dimensional Turing patterns consisting of rhombic arrays of rectangles and hexagonal is obtain by singular perturbations theory. Finally, by the method of computer simulation of the model, we describe two different patterns.

1. Introduction

Generally, most dynamics in chemical reaction are nonlinear. However, when systems approach thermodynamics steady state, their kinetics behavior are studied approximately by linear non-equilibrium thermodynamics. When systems are far away thermodynamics steady state, sometimes, nonlinear effects become principal factor of dynamics behavior, the nonlinear behavior couple by diffusive linear behavior. The coupling initially may give rise to the spontaneous appearance of the order and chaotic pattern. Pattern formation in biological contexts has been studied by Nagorck et al. [1] and Cruywagen and Murray [2] by coupling together a reaction diffusion system governing the concentration of a pair of morphogens and a cell traction model of the elastic properties of the dermis. Each mechanism is capable of generating spatial patterns on its own, and the coupling leads to the formation of spatial patterns that are more complicated than those seen in either model separately and which are closer to those actually observed. Showalter and coworkers [3] have performed experiments and mathematically modeled the effect of coupling on traveling waves in chemical systems. In these experiments the reacting medium is the Belousov–Zhabotinskii (BZ) reaction in which an inhomogeneous medium is created by removing the ferroin (catalyst) from the bulk solution and immobilizing it in a Nafion ion-exchange membrane. This is then bathed in a solution of the remaining reactants and thus the reaction is restricted to the interfaces between the membrane and the bulk solution.

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Coupling occurs due to diffusive transport through the membrane of nonionic species such as $HBrO_2$ and gives rise to the formation of new complex spatiotemporal structures.

The feature common to nearly all isothermal oscillatory reactions is autocatalysis. Autocatalysis is shared not only by the Belousov–Zhabotinskii (BZ) reaction, a wide range of halide-based oscillations, and the arsenite plus iodate reaction but by iodate-arsenous acid reaction. All mechanistic schemes include autocatalytic steps (or autocatalytic combinations of elementary steps). Forms of these schemes need to given a simpler prototypes in a logically ordered way.

Singularly perturbed equations are often used as mathematical models describing processes in physics, chemical kinetics, and mathematical biology, and they often arise during investigation of applied problems of technology and engineering. As a class of elementary reaction that can replicate themselves and produce complex chemical morphogenesis and pattern, cubic autocatalysis has formed a great threat to human society and been shown to form the key steps in a number of more complex reaction schemes, see, for example, [4–6]. The model of cubic autocatalysis offer scientist in such far ranging fields as biology, physics, social science and mathematics, an important example of the phenomenon of critical behavior on the pattern and perturbed model. Gafiychuk and Datsko [7] investigate pattern formation in a fractional reaction–diffusion system. By the method of computer simulation of the model of excitable media with cubic nonlinearity they are able to show structure formation in the system with time and space fractional derivatives.In many epidemiological models, the corresponding incidence rate is nonlinear with respect to the numbers of susceptible and infective individuals. The mechanism of cubic autocatalysis is one of the main mechanism of forming nonlinear term of biological model. In [8], Yang et al. include stochastic perturbations into SIR and SEIR epidemic models with saturated incidence and investigate their dynamics according to the basic reproduction number. In [9], certain two component reaction diffusion systems on a finite are known to possess mesa (box-like) steady state patterns in the singularly perturbed limit of small diffusivity for one of the two solution components. In [10], Stochastic partial differential equations are introduced for the continuum concentration fields of reaction diffusion systems. Spatially adaptive stochastic numerical methods are developed for approximation of the stochastic partial differential equations. Pattern formation in evolutionary dynamics has been studied as an example of phenomena inherent to many natural system. It is well known that reaction–diffusion models are used to study self-organization phenomena in physical, chemical and biological systems. In [11], two SIR epidemic models with different patterns of recruitment and difference in immunity are investigated. In [12], stable Turing patterns are presented in the simplest reaction diffusion system containing a single autocatalitic step in a continuously fed unstirred reactor. In [13], applying two types of Lyapunov functional techniques to an SIRS epidemic model with graded cure and incomplete recovery rates, Murroya et al. establish complete global dynamics of the model.

In this paper, we consider the prototype chemical reaction scheme based on the cubic autocatalator ***which is proposed in*** [5]

$$P \to A\,(rate\ k_0 p), \tag{1}$$

$$A + 2B \to 3B\,(rate\ k_1 a b^2), \tag{2}$$

$$B \to C\,(rate\ k_2 b), \tag{3}$$

where a and b are the concentrations of the species A and B respectively, and k_1, k_2 are the rate constants (both reaction are assumed to be isothermal). Here we supposed that the reaction is taking place inside a closed system with the reactant A now being replenished by the slow decay of a precursor P, where p is the concentration of P and k_0 is the rate constant. The reaction diffusion system has been discussed in some detail for the strip spatial variation patterns in one dimension system. The formation of spatial patterns is considered in [14], two types of one dimension patterns occur, standing wave patterns arising out of Hopf bifurcations, together with steady wave patterns arising out of pitchfork bifurcations. In [15], we consider spatiotemporal structures and bifurcation arising in two identical cells, which are governed by higher autocatalator kinetics and coupled via diffusive interchange of autocatalyst.

In the present paper, we consider a coupled system whereby two identical cells are coupled via the diffusion interchange of the autocatalyst B. In [16], we have considered one dimensional spatial diffusion model and given one strip pattern. Further, we consider the spatial variation to two dimensions and measure distance in these dimensions with the coordinate x and y. The vessel is taken to have length 1 and width 1 with impermeable boundaries at the ends $x = 0, 1$ and $y = 0, 1$. The equations are as in [14] but now the effect of spatial variation y included, namely

$$\begin{cases} \frac{\partial a_1}{\partial t} = D\nabla_2^2 a_1 + \mu - a_1 b_1^2, \\ \frac{\partial b_1}{\partial t} = D\delta\nabla_2^2 b_1 + a_1 b_1^2 - b_1 + \beta(b_2 - b_1), \\ \frac{\partial a_2}{\partial t} = D\nabla_2^2 a_2 + \mu - a_2 b_2^2, \\ \frac{\partial b_2}{\partial t} = D\delta\nabla_2^2 b_2 + a_2 b_2^2 - b_2 + \beta(b_1 - b_2). \end{cases} \tag{4}$$

Here, $\nabla_2^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, D and δ are constant diffusion coefficients and t is the time, and β is the dimensionless coupling parameter, (a_1, b_1) and (a_2, b_2) are dimensionless concentration in cells 1 and 2. Consider the two dimensional domain spatial region defined by $B = \{(x, y)|0 < x < 1, 0 < y < 1\}$, whose rectangular boundary, we denote by ∂B. In addition to Eq. (4), we have the boundary conditions associated with the impermeable walls, that

$$\frac{\partial a_i}{\partial n} = \frac{\partial b_i}{\partial n} = 0 (i = 1, 2), \quad (x, y) \in \partial B \tag{5}$$