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Nonlinear bending and post-buckling of extensible microscale beams based on modified couple stress theory

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ABSTRACT

This study proposes a computationally efficient approach to nonlinear bending and thermal post-buckling problems in Euler–Bernoulli microbeams based on modified couple stress theory under geometrically accurate relationships. The governing equations, which consider the size effect and the axis extensibility, are formulated via the equilibrium of an infinitesimal element. The proposed model, which encompasses the size-independent and Von Kármán nonlinear theory, is solved using the shooting technique after transformation into a two-point boundary value problem. The proposed method was validated based on comparisons with several case studies using existing simulations. The influences of the length scale parameter and the Poisson ratio on the bending and thermal post-buckling behaviors of microbeams are discussed in detail. The numerical results show that the intrinsic size dependency of the material and the Poisson ratio make the microbeam behave in a relatively stiff manner, thereby leading to smaller deformations and greater increases in the buckling temperature.

1. Introduction

Microstructures have extensive applications in micro/nano-scale devices and systems [1]. The study of the bending, buckling, and vibration behaviors of microstructures is of practical importance for understanding mechanical responses. For example, microbeams that are used as microstructural components are subjected to thermal loads or compressive loads on a regular basis, thus they are more susceptible to buckling instability. Hence, it is necessary to predict the buckling characteristics of these beams.

Experimental investigations have indicated that structures at the micron- and sub-micron scales exhibit many size-dependent mechanical behaviors [2]. The typical experiments reported in this area include micro-torsion tests of thin copper wires by Fleck et al. [3], micro-bending tests of thin nickel beams by Stolken et al. [4], micro-bending tests of epoxy polymeric beams by Lam et al. [5], and micro-bending tests of polypropylene cantilevers by McFarland et al. [6]. These experimental studies have shown that classical theory underestimates the stiffness of microstructures. Thus, the inclusion of length scale parameters in constitutive models has become inevitable. A mathematical model based on conventional elasticity theory is insufficient for describing these size effects due to the lack of a material length scale parameter in their formulations, thus they must be appropriately modified. Recently, some nonclassical continuum theories have been developed where constitutive equations introduce other material parameters in addition to the classical elastic parameters.

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There are three main promising types of size-dependent theories, i.e., higher order gradient theories [7], nonlocal elasticity theories [8,9], and couple stress theories [10,11], where the latter has attracted the most interest.

Mindlin and Tiersten [10] and Koiter [11] are pioneering researchers who elaborated the couple stress theory, including two classical and two additional material length scale parameters for isotropic elastic materials. Yang et al. [12] reduced the additional length scale parameters to only one and produced a modified couple stress theory to explore the size effects, where this modified version simplifies the complex relations in couple stress theory and it has encouraged many researchers to analyze correlations with microscale structures.

Utilizing the modified couple stress theory, some Euler–Bernoulli [13–18] and Timoshenko [19–22] microbeam models have been constructed for bending [13–15,19–21], buckling [16], postbuckling [14], linear [18–20,22], and nonlinear [14,17,21] vibration problems related to microbeams. For example, Park and Gao [13] analyzed the static mechanical properties of an Euler–Bernoulli beam and interpreted a bending test using an epoxy polymeric beam. This model was employed by Kong et al. [18] to study the free vibration of microbeams. Xia et al. [14] presented a Euler–Bernoulli beam model that included the von Kármán nonlinear strain, but where the coupling between axial and transverse displacements was neglected. The bending [15] and buckling [16] of microscaled beams based on both the modified couple stress theory and the modified strain gradient elasticity theory were studied by Akgöz and Civalek. Ma et al. [19] and Asghari et al. [21] formulated Timoshenko beam models in geometric linear and nonlinear domains, respectively, and assessed the size-dependent static and free vibration behaviors. Wang et al. [17] studied the free vibration using the Von Kármán geometrical relation, where a mathematical model was established in partial differential form based on the Hamilton principle and it was solved by the shooting method after the separation of variables and executing the Kantorovich technique. Homogeneous and functionally graded beam models have been formulated based on the modified couple stress theory [20,22–25]. For example, Asghari et al. [20] generalized the work of Ma et al. [19] to functionally graded beams. Ke and Wang [22] investigated the dynamic stability of functionally graded microbeams based on Timoshenko beam theory. Nateghi et al. [23] presented a buckling analysis of functionally graded microbeams based on the modified couple stress theory where the effects of the material length scale parameter, beam thickness, and the Poisson ratio on the size-dependent buckling load were illustrated numerically. In another study, they discussed thermal effects on size-dependent buckling and the free vibration behavior of functionally graded microbeams using two different beam theories [24]. Reddy [25] developed microstructure-dependent Euler–Bernoulli and Timoshenko beam theories for functionally graded microbeams, which considered geometrical nonlinearities by applying von Kármán strain tensor, where an analytical solution was presented for a linear case using the Navier procedure. In addition, the modified couple stress theory has been applied to plates in studies of the static deflection [26] and free vibrations [27,28] of microplates. These previous studies suggest that the aforementioned theory is effective in providing insights into size effects in microscale structures.

For thin beams that are used widely in micro- and nanoelectromechanical systems, the effects of geometric nonlinearity are significant and they should be considered when studying their static and dynamic characteristics [29–32]. In addition, predicting the post-buckling configurations or equilibrium paths of the beam necessitates the introduction of geometrical nonlinearities into the model, although critical buckling loads can generally be deduced from a linear analysis. Thus, the analysis of size-dependent microbeams in nonlinear ranges is essential. Previously, Xia et al. [14], Wang et al. [17], Asghari et al. [21], and Reddy [25] focused on the nonlinear static bending, post-buckling, and free vibration of nonclassical microscale beams, where the linear expression of curvature was retained and the von Kármán-type strain displacement relation was introduced.

In the current study, we investigated the bending and buckling characteristics of Euler–Bernoulli microbeams by considering geometrically accurate relationships and small-scale effects within the framework of the modified couple stress theory. The size-dependent governing differential equations are determined based on the equilibrium of an infinitesimal element and they are solved numerically via the shooting method. To establish the bending and post-buckling equilibrium configuration in an efficient manner, the central deflection of the microbeam is treated as a control parameter whereas the transverse load in the bending case and the temperature increase in the buckling case are unknown functions. We obtained numerical results to demonstrate the effects of the material length scale parameter, temperature increases, and the Poisson ratio on the size-dependent nonlinear bending and post-buckling behaviors of microbeams.

2. Governing equations for the microbeams

As shown in Fig. 1(a), a microbeam with an initial length L, cross-sectional area A, and moment of inertia I, which is made of an elastic material with a Young's modulus E and Poisson ratio ν, is modeled as an extensible Euler–Bernoulli beam. A rectangular coordinate system (x, y, z) is located at the left-most end of the beam where the x, y, and z axes are taken along the length, width, and height of the beam, respectively. The centroid of each section is assumed to lie on the plane $z = 0$.

2.1. Preliminaries

The modified couple stress theory [12] has an advantage compared with the classical couple stress theory because it only includes a single scale parameter. In this theory, the constitutive equations in the thermal environment can be written as

$$\boldsymbol{\sigma} = \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} - \alpha(3\lambda + 2\mu)\Delta T\boldsymbol{I}, \tag{1}$$