Applied Mathematical Modelling 39 (2015) 255–269

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Tear film dynamics with evaporation, osmolarity and surfactant transport

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ARTICLE INFO

Article history:
Received 5 November 2012
Received in revised form 17 March 2014
Accepted 15 May 2014
Available online 2 June 2014

Keywords:
Tear film rupture
Thin film dynamics
Evaporation
Osmolarity
Solutal Marangoni effect

ABSTRACT

In this article we develop a model for the evaporation and rupture of the tear film. The tear film is generally considered a multi-layer structure which we simplify to a single layer in our modeling. We examine how well the floating lipid layer can be approximated by a mobile insoluble surfactant monolayer in the context of lubrication theory with film rupture "breakup" in the tear film literature. This model includes the effects of surface tension, insoluble surfactant monolayer transport, solutal Marangoni effects, evaporation, osmolarity transport, osmosis and wettability of corneal surface. Evaporation is hypothesized to be dependent on pressure, temperature and surface concentration at the surface of the film. A focus of this paper is to study the competition between the effect of increasing surfactant concentration to (1) slowing down evaporation and (2) lowering surface tension. The solutal Marangoni effect, for local increases in surfactant concentration, can induce local thinning and this effect always seems to dominate the reduction in thinning rate due to evaporation in our model. It also seems to eliminate any localized area of increased evaporation due to reduced surfactant concentration. Osmolarity in the tear film increases because water lost to the average evaporation rate and to a lesser extent by flow inside the film. The presence of van der Waals conjoining pressure is only significant when osmosis is very small or absent. The model predicts that the Marangoni effect coupled with evaporation can determine the location of first breakup; it also agrees with another model of breakup that predicts elevated osmolarity when breakup occurs.

1. Introduction

The ocular tear film is critical for good vision. When functioning properly, the tear film moisturizes, transports waste and provides a smooth optical surface [1]. It comprises multiple layers [2,3]. There is an anterior layer of lipids [4–6] that is tens of nanometers thick on the average. The lipids float upon a primarily aqueous layer [1] which is in turn on a mucin-rich region at the corneal surface [7–9]. The precorneal tear film is a total of a few microns thick in the center of the cornea after a blink [10–12] and has a considerably thicker meniscus around the lid margins [13–15] where the tear film climbs up the wettable part of the eyelids. The tear film must reform itself rapidly with each blink, forming the relatively smooth optical surface for clear vision; most motion dies out after a second or two, but then evaporation and tear film break up (rupture) may become significant on longer time scales [6,16,17]. Evidence for evaporation of water comes from measurements of the

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http://dx.doi.org/10.1016/j.apm.2014.05.006
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mass lost from evaporation [18], and from thinning rates [19]. According to [18], thinning rate measurements typically yield larger evaporation rates (when converted to this form) compared to those found from methods using goggles. In [19] an interferometric technique is used to measure the thinning rates with and without air tight goggles where the lack of airflow and water vapor diffusion stopped the evaporation: this indicates that evaporation is a major contribution to tear film thinning. Losing water from the tear film increases the saltiness of tears; the concentration of ions that induce osmosis is called the osmolarity. Chronically increased osmolarity from high evaporation rates is thought to be a significant factor in the development of dry eye [20].

The tear film's complex structure requires simplification to make it amenable to mathematical modeling. A sample of tears combining all of its components is mildly shear thinning [21] and weakly elastic [22]. Recent measurements of meibomian lipids alone show that they may have significant elasticity [23] at room temperature but the elasticity diminishes markedly at or above 32 °C [24]. The structure of the lipid layer appears to have some order, though the extent of the order is still a matter of debate [25,26]. The contribution of the lipid layer to the overall rheology and properties of tears at *in vivo* temperatures is yet to be understood. The apparent surface tension of the tear-air interface and other interfacial properties are affected by surface active polar lipids [27,28] that are thought to occupy the lipid-aqueous interface and to be insoluble in the aqueous layer. The polar lipids can cause upward motion of the tear fluid after a blink [29,30]. However, the apparent surface tension of tears appears to be affected by proteins and other molecules as well [31,32]. The effectiveness of the tear film lipid layer in reducing evaporation is variable and difficult to understand [33], though there is evidence for the close link between lipid dynamics and tear film breakup [34,6,35–37]. The literature about the lipid layer is extensive, and we only give some recent references here.

Theoretical models of tear film dynamics have recently been reviewed [38]; a brief review is paraphrased here from [39]. The tear film is commonly assumed to be Newtonian, and the underlying substrate (the cornea) is assumed to be flat [29,39]. Mathematical studies have incorporated a variety of important effects: surface tension [40–43]; polar lipid surface concentration gradients causing the Marangoni effect [29,44–46]; evaporation [47,48,6]; wettability of the corneal surface via van der Waals terms [49–51,48]; motion of the eye lids in one dimension [44–46,52–56]; heat transfer from the eye posterior to the tear film [57]; and the shape of the eye opening [58,59]. These effects may all contribute in different regions of the eye and at different times in the blink cycle.

Of particular interest here are [45,46,56]. These papers treat the lipid layer as an insoluble surfactant layer, and only in the last is there evaporation from the tear film. All of them treated the opening phase of the blink and subsequent interblink. The properties of the insoluble surfactant could be chosen to fit upward drift of the lipid layer observed during the first 2 s or so of the interblink. These models fit the blink and early part of the interblink well, when the surface active nature of the lipid layer is important [30,6]. We aim to explore how well a similar model approximates tear film breakup with evaporation included; [56] did not study tear film breakup, and used a different evaporation model.

Previous theoretical efforts in modeling tear film breakup have provided some insight, and have left some important phenomena to explain and clarify. A mechanism describing the tear film breakup was first proposed by [60] based on a non-wettable corneal surface and which was later supported by some experimental evidence by [61]. Following [60], a non-Newtonian model was developed for tear film break up due to dewetting van der Waals forces emphasizing the dynamics of a mucus layer at the ocular surface. In the ocular surface community, the corneal surface has more recently been viewed as wettable (when healthy), and while a theoretical treatment of a separate mucus layer may be justifiable, it is not seen experimentally; see [38] for a more complete discussion. The aim of current study is to study a mathematical model that illuminates how well a surfactant monolayer model can mimic the expected dynamics of the tear film lipid layer. The model mimics the evaporation barrier by slowing the thinning rate for increasing surfactant concentration. Increasing surfactant concentration lowers surface tension as well. We examine the competition of these two effects in the context of tear film breakup. We also study the effect of the resulting dynamics on the osmolarity (saltiness) inside the film.

The osmolarity is the ion and solute concentration causing osmosis from the ocular surface. It is thought to be a critical variable in the development of dry eye disease [20]. The osmolarity is thought to become elevated as a result of evaporation of water from the tear film [62], which in turn places stress to the ocular surface [38]. We do not treat the biological effects of osmolarity; we will treat the ocular surface as a perfect semi-permeable membrane [63,38], and link the properties to recent experimental determinations of the permeability [64]. The development of the transport equation for the osmolarity follows the approach of [65] as described in [38]. We will compute the osmolarity as well as the other variables in the model. During the completion of this work, another breakup model was completed that treated the lipid layer as a fixed evaporation distribution with no surfactant [66] with diffusion resisting evaporation through a stationary lipid layer and transport through the air outside the film. That work finds elevated osmolarity in the breakup region and although there is a different mechanism for breakup here, the model in this paper predicts this as well.

In this paper, we study a simplified model of break up of the tear film. We neglect the menisci that form at the lid margins. We focus on the local dynamics away from the mensici that would be unaffected by them [37]. Although the tear film is a multilayer structure (e.g. [4,8,9]) we simplify the lipid layer to an insoluble surfactant on an underlying aqueous layer. The break up of the tear film is driven by evaporation, while van der Waals forces cause a conjoining pressure that keeps the tear film thickness from reaching zero [48,57]. We include the transport of osmolarity inside the tear film and osmosis from the cornea via a boundary condition at the base of the film. Both osmosis and van der Waals forces act to oppose thinning due to evaporation. We study the effect of osmosis parametrically for different permeabilities of the cornea. The structure of