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On the numerical determination of eigenvalues/eigenvectors using a high regularity finite element method

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ABSTRACT

This study investigates the numerical prediction of eigenvalues/eigenvectors in heat conduction transfer problems based on approximation spaces constructed using a high regularity Hermite finite element method (H-FEM). Special attention is given to the number of reliable numerical eigenvalues that can be estimated using this approach. The shape functions are constructed on quadrilateral elements by the tensor product of the generalization of Hermite functions in one-dimensional space. The numerical results obtained for selected eigenvalue/eigenvector problems using the H-FEM are compared with available analytical solutions and numerical solutions obtained using the high-order Lagrange FEM, as well as with the B-splines FEM. The numerical results demonstrate the superior accuracy of the H-FEM in predicting high order modes, thereby reducing the appearance of spurious branches in the spectrum.

1. Introduction

The eigenvalue/eigenvector problem has been explored widely, especially for problems related to obtaining the natural frequencies and modes of a structural component. Obtaining a high percentage of numerically approximate eigenvalues within an acceptable accuracy limit is still an open research topic, and several methods are proposed to address this problem appear each year. This limitation is even more apparent when low regularity approximation spaces are employed in the numerical analysis, which directly reflect the propagation problems caused by excitation pulses with short durations. The limitations encountered in the determination of eigenvalues/eigenvectors in solid mechanics also affect other physical phenomena, such as elliptical and parabolic thermal problems.

The C^0 class of the finite element method (FEM) has been demonstrated to have the limitations mentioned above, as described in several previous studies. This makes it very difficult to represent some types of problems where the appropriate determinations of eigenvalues and higher order modes are essential. To satisfactorily obtain the higher frequencies in these cases, it is necessary to approach the problem with an excessive number of degrees of freedom because only a small percentage of the eigenvalues and the modes are numerically approximated within an acceptable accuracy. For the sake of argument and illustration, we present the results obtained using the low regularity FEM (C^0), but we consider the p adaptivity during

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the analysis of the natural frequencies of an elastic rod (with mass density ρ, Young's modulus E, cross-section area A, and length L), which is clamped at one end, $u(0) = 0$, and submitted to a condition $\frac{du(x)}{dx}\Big|_{x=L} = 0$ at the other end, as one of the simplest problems in mechanics. The elliptical eigenvalue problem of this model can be stated as follows. Determine the pairs $(\omega_n, u_n(x))$ such that

$$\frac{d^2u(x)}{dx^2} + \omega^2 \frac{\rho}{AE} u(x) = 0, \quad \forall x \in (0, L). \tag{1}$$

The solution of Eq. (1) is given by an infinite set of countable pairs $(\omega_j, u_j(x))$, $j = 1, 2, \ldots, \infty$ where $0 < \omega_1^2 < \omega_2^2 <, \ldots, < \omega_{j-1}^2 < \omega_j^2 < \omega_{j+1}^2 <, \ldots$ with

$$\omega_j = \frac{(2j-1)\pi}{2L\sqrt{\frac{\rho}{AE}}}, \quad j = 1, 2, \ldots, \tag{2}$$

$$u_j(x) = C \sin\left(\sqrt{\frac{\rho}{AE}}\omega_j x\right), \tag{3}$$

as shown in Fig. 1. Note that there is a significant loss of accuracy for high modes when the low regularity FEM is used with high-order elements. This loss of accuracy is indicated by the difference from the exact result, $\frac{\omega_h}{\omega} = 1$, in the normalized approximated curves, as well as by the jump in the frequency spectrum, which defines the acoustic and optical branches. The degree p of the finite element space is higher when the error of the high modes is also greater. The results presented in Fig. 1 are intriguing. The authors of [1] suggest that they were the first to observe/publish this effect in 2007.

Previous studies [1–5] considered the construction of high regularity approximation spaces based on an isogeometric analysis. In these studies, more accurate results were obtained compared with those produced using low regularity FEM for the determination of the natural frequencies and modes of some problems, especially for solid mechanics problems. The k-method proposed in [2] uses the isogeometric methodology to build high-order approximation spaces with the desired/dependent regularity. The results obtained using this method are more accurate than those with the low regularity FEM in the analysis of the natural frequencies of an elastic rod problem. The normalized value curves of the natural frequencies are smooth and the accuracy increases with the polynomial order and the regularity of the approximation space, as reported previously [3,4,1]. On first glance, one might conclude that the low efficiency of the FEM numerical approximations of elliptic eigenvalue problems is related to the low regularity of the approximation spaces. However, a well known result ensures that since the mathematical features of a finite element space $\mathcal{U}^h$ are known, i.e., the degree of continuity k, polynomial degree p, and a mesh parameter h, then the approximate eigenvalues λ_i^h, $i = 1, 2, \ldots, N$, are bounded, and a proof is provided in [6], as follows.

Theorem 1. *A priori error estimates of eigenvalues: If $\mathcal{U}^h$ is a finite element space of degree p, then there is a constant c such that the approximate eigenvalues are bounded for small values of h by*

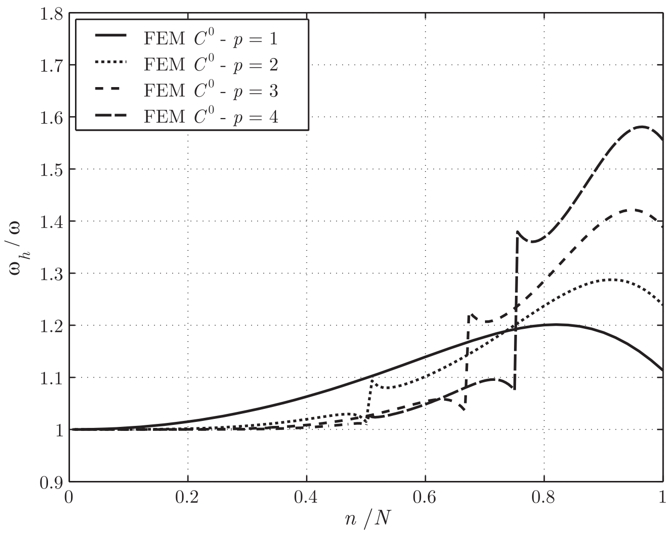

Fig. 1. C^0 p-FEM normalized natural frequencies versus mode number results for the longitudinal vibrations of an elastic rod, where n is the mode number, N is the total number of degrees of freedom considered in the analysis, ω_h is the approximated result, and ω is the analytical result.