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Analytical study on a two-dimensional Korteweg–de Vries model with bilinear representation, Bäcklund transformation and soliton solutions

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ABSTRACT

With symbolic computation, Bell-polynomial scheme and bilinear method are applied to a two-dimensional Korteweg–de Vries (KdV) model, which is firstly proposed with Lax pair generating technique. Bell-polynomial expression with one auxiliary independent variable is derived and transformed into bilinear form. According to the coupled two-field conditions between the primary and replica fields, Bell-polynomial-typed Bäcklund transformations (BTs) are constructed and converted into the bilinear ones. Finally, soliton solutions of the two-dimensional KdV model are obtained (via solving the bilinear representation and BT, respectively) and compared. Such associated integrable properties as bilinear representation, BT (especially auxiliary-independent-variable-involved Bell-polynomial-typed ones constructed in this paper) and soliton solutions (especially the multi-soliton ones) may be useful for further study on other two-dimensional KdV and KdV-typed models.

1. Introduction

In dealing with nonlinear models of mathematical physics [1–25], there exist types of numerical [7,8,18] and analytical approaches [5,6,9,17,20–24]. For example, a meshless based numerical technique has been used to find traveling solitary wave solution of Boussinesq equation [8], and the multiple exp-function algorithm has been used to search for three-wave solutions [20,22]. The Bell-polynomial scheme [1–6] is found to be a direct and systematic method, among others. Procedure of the Bell-polynomial scheme mainly includes the following steps [5,6]: $(\mathcal{A})$ derive the Bell-polynomial expression of the original nonlinear model with the help of its invariance under the scale transformations; $(\mathcal{B})$ decompose the homogeneous constraint (or, two-field condition) between the primary field and a replica one suitably to construct the Bell-polynomial-typed Bäcklund transformation (BT) generally expressed in terms of the linear combination of Bell-polynomials and their derivatives; $(\mathcal{C})$ linearize the Bell-polynomial-typed BT through the Hopf–Cole transformation to obtain the corresponding Lax pair.

There exists a close connection between the Bell-polynomial scheme and Hirota bilinear method [2,5,6,19]. Generally speaking, the Bell-polynomial expression in step $(\mathcal{A})$ can be cast into the bilinear equation of the original nonlinear model,

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and the Bell-polynomial-typed BT in step ($\mathcal{B}$) into the bilinear BT correspondingly. Note that a class of generalized bilinear operators have been introduced [23], and it has explored when the linear superposition principle can be applied to the resulting generalized bilinear differential equations; while in Ref. [24], the resulting bilinear differential equations are characterized by a special kind of Bell polynomials and the linear superposition principle is applied to the construction of their linear subspaces of solutions. Moreover, the trilinear differential operators have been introduced and used to create trilinear differential equations [21], while the resulting trilinear differential operators and equations are characterized by the Bell polynomials, and the superposition principle is applied to the construction of resonant solutions of exponential waves.

Within the framework of Bell-polynomial manipulation, some nonlinear models have been addressed, such as the Burgers–Hopf hierarchy, potential Korteweg–de Vries (KdV), modified KdV, potential Sawada–Kotera, sine–Gordon, Boussinesq equations and Ablowitz–Kaup–Newell–Segur system [2,3]. Note that, for some other nonlinear models, one or more auxiliary independent variable(s) have been introduced in the Bell-polynomial manipulation, e.g., the shallow water wave equation, Lax fifth-order KdV equation, (2 + 1)- and (3 + 1)-dimensional breaking soliton equations, sine–Gordon equation, Tzitzeica equation, Liouville equation and an alternative to the Boussinesq equation (see Refs. [4,6]).

In the present paper, we will employ the Bell-polynomial scheme and bilinear method to investigate bilinear representation, BT and soliton solutions for the following two-dimensional KdV model [10]:

$$u_t + 6\,u\,u_x + u_{xxx} + 4\,u\,u_y + u_{xxy} + 2\,u_x \int u_y\,\mathrm{d}x = 0, \tag{1}$$

which was firstly proposed in Ref. [10] with Lax pair generating technique. The detailed Lax pair associated with Eq. (1) can be derived correspondingly in virtue of Eq. (5) in Ref. [10]. Meanwhile, exact solutions of Eq. (1) have been studied by means of the singular manifold method, BT in terms of the singular manifold has been obtained, and localized structures have also been investigated. To our knowledge, Bell-polynomial manipulation and bilinear calculation have not been performed on Eq. (1).

With symbolic computation [9,12–16], some integrable properties of Eq. (1) will be studied as follows: In Section 2, the Bell-polynomial expression of Eq. (1) will be derived with the help of its invariance under the scale transformations, where note that an auxiliary independent variable will be introduced; In Section 3, the Bell-polynomial expression will be converted to achieve the bilinear representation, which will be used to calculate soliton solutions of Eq. (1); In Section 4, the Bell-polynomial-typed BT and bilinear BT with an explicit solution will be constructed. Finally, Section 5 will be our conclusions.

2. Bell-polynomial expression

With the assumption that z is a C^∞ function of the variable x and $z_r = \partial_x^r z$ $(r = 1, 2, \ldots)$, the Bell polynomials [1–3] are defined as follows:

$$Y_{nx}(z) \equiv Y_n(z_1, \ldots, z_n) = e^{-z}\partial_x^n e^z = \sum \frac{n!}{c_1! \cdots c_n!(1!)^{c_1} \cdots (n!)^{c_n}} z_1^{c_1} \cdots z_n^{c_n}, \tag{2}$$

where the sum is to be taken over all partitions of $n = c_1 + 2c_2 + \cdots + nc_n$, for example,

$$Y_1 = z_1, \quad Y_2 = z_2 + z_1^2, \quad Y_3 = z_3 + 3z_1z_2 + z_1^3, \ldots. \tag{3}$$

With the new dependent variables W and V, the binary Bell polynomials and two-dimensional binary Bell polynomials respectively take the following forms as

$$\mathcal{Y}_{nx}(V, W) \equiv Y_n(z_1, \ldots, z_n)\Big|_{z_m = \begin{cases} V_{mx} & \text{if } m \text{ is odd} \\ W_{mx} & \text{if } m \text{ is even} \end{cases}} \tag{4}$$

and

$$\mathcal{Y}_{mx,nt}(V, W) = Y_{mx,nt}(z_{r,s})\Big|_{z_{r,s} = \begin{cases} V_{r,s} & \text{if } r+s \text{ is odd} \\ W_{r,s} & \text{if } r+s \text{ is even} \end{cases}} \tag{5}$$

Introducing the new field $q = W - V$, we can find that the members of the following family of even-order $\mathcal{Y}$-polynomials, named as,

$$\mathcal{P}_{2nx}(q) = \mathcal{Y}_{2nx}(0, q), \tag{6}$$

obey a new partition rule obtained by restricting the Bell recipe to even part partitions:

$$\mathcal{P}_0 = 1, \quad \mathcal{P}_{2x}(q) = q_{xx}, \quad \mathcal{P}_{xt}(q) = q_{xt}, \tag{7}$$

$$\mathcal{P}_{4x}(q) = q_{4x} + 3q_{xx}^2, \quad \mathcal{P}_{3x,t}(q) = q_{xxxt} + 3q_{xx}q_{xt}, \tag{8}$$