Applied Mathematical Modelling 39 (2015) 3366–3379

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Green's function for uniform Euler–Bernoulli beams at resonant condition: Introduction of Fredholm Alternative Theorem

Seyed Mojtaba Hozhabrossadati [a,*], Ahmad Aftabi Sani [a], Bahman Mehri [b], Masood Mofid [c]

[a] Department of Civil Engineering, Mashhad Branch, Islamic Azad University, Masshad, Iran
[b] Department of Mathematics, Sharif University of Technology, Tehran, Iran
[c] Department of Civil Engineering, Sharif University of Technology, Tehran, Iran

ARTICLE INFO

Article history:
Received 16 November 2013
Received in revised form 6 October 2014
Accepted 24 November 2014
Available online 4 December 2014

Keywords:
Fredholm Alternative Theorem
Modified Green's function
Resonance
Euler–Bernoulli beam

ABSTRACT

This paper deals with the dynamic analysis of Euler–Bernoulli beams at the resonant condition. The governing partial differential equation of the problem is converted into an ordinary differential equation by applying the well-known Fourier transform. The solution develops a Green's function method which involves establishing the Green's function of the problem, applying the pertinent boundary conditions of the beam. Due to the special conditions of the resonant situation, a significant obstacle arises during the derivation of the Green's function. In order to overcome this hurdle, however, the Fredholm Alternative Theorem is employed; and it is shown that the modified Green's function of the beam may still be achievable. Furthermore, the necessary requirement so that the resonant response will be found is introduced. A special case which refers to a case in the absence of resonance is also included, for some verification purposes.

1. Introduction

Great emphasis has been placed on the analytically based methods of solving the governing differential equations in the continuous systems problems up to now. This has been reflected in the number of applied mathematical texts oriented towards particular analytical methodologies, in particular, the Green's function method [1–4]. Green's function method has received considerable attentions in the dynamic analysis of continuous systems among researchers. Kukla [1] used the Green's function method in the free vibration analysis of a beam with intermediate elastic supports. He also gave the Green's functions for three types of boundary conditions: pinned–pinned, sliding-sliding and free-free. Green's functions for beams of several common boundary conditions have been tabulated in Mohamad's work [2]. In addition, natural frequencies and mode shapes of beams with intermediate attachments were presented in his work. Moreover, Lueschen and Bergman [3] presented the Green's function for uniform Timoshenko beams as well as Euler–Bernoulli beams with and without a constant axial preload. Foda and Abduljabbar [4] used a Green's function approach to determine the response of a simply supported Euler–Bernoulli beam of finite length subject to a moving mass.

After that, Abu-Hilal [5] utilized a Green's function method for determining the dynamic response of Euler–Bernoulli beams subjected to distributed and concentrated loads. In his work, Green's functions for various beams with different elastic boundary conditions were thoroughly investigated. The solution of free vibration problem of stepped beams under

* Corresponding author.
E-mail address: sm.hozhabrossadati@yahoo.com (S.M. Hozhabrossadati).

http://dx.doi.org/10.1016/j.apm.2014.11.038
0307-904X/

axial loads using Green's function method is given by Kukla [6]. In that study, the exact transcendental frequency equation was solved by using a numerical method and it was shown that the accuracy of the numerically obtained eigenfrequencies improves as the step number of the stepped beams increases. It is worthwhile to mention that Mehri, Davar and Rahmani [7] presented a technique based on the dynamic Green's function for the analysis of Euler–Bernoulli beams with different boundary conditions under a moving load.

In the vicinity of the works on the dynamic analysis of beams, some researchers attempt to present solutions based on the static Green's function of the mentioned beams. For instance, Failla [8] proposed a solution for Euler–Bernoulli arbitrary discontinuous beams using the static Green's function.

As mentioned previously, research to date on the Green's function has resulted in the publication of a great deal of articles in a wide range of journals with backgrounds in mechanics of solids. However, no extensive, detailed treatment of the behavior of the Green's functions at the resonant condition has been reported. The objective of this paper is to fill this gap, so that readers can obtain a sound knowledge on this topic.

In this paper, consideration is given to the solution of forced vibration of simply supported Euler–Bernoulli beams in the presence of resonance. The analysis is performed in the frequency domain by means of the Fourier transform. The solution is based on the Green's function method and consequently, closed-form results are obtained. It should be noted that the term Green's function refers to a function, associated with a given boundary value problem, which appears as an integrand for an integral representation of the solution to the problem [9]. From a physical point of view, the Green's function is the response of a beam (displacement) to a unit impulse.

During the process of constructing the Green's function of the problem, a contradiction occurs, due to the special situation of the resonance. To remedy this, the Fredholm Alternative Theorem is employed and the modified Green's function of the problem is found. By this mean, the mentioned contradiction will be eliminated, and the resonant response of the beam will be achieved. In the next section, for some verification purposes, a special case is also included which refers to the problem in the absence of resonance. Finally, some numerical examples are given and the results are discussed.

2. Problem formulation

The partial differential equation which governs the transverse vibration of a uniform thin beam with finite length based on the Euler–Bernoulli theory is of the form

$$\mathrm{EI}\frac{\partial^4 U(x,t)}{\partial x^4} + m\frac{\partial^2 U(x,t)}{\partial t^2} = Q(x,t), \tag{1}$$

where $U(x,t)$ and $Q(x,t)$ are the functions of transverse displacement and transverse loading, and EI and m are two constants which denote the flexural stiffness and the mass per unit length of the beam, respectively. Applying the well-known Fourier transform with respect to t to both sides of Eq. (1), yields

$$\mathrm{EI}u''''(x,\omega) - m\omega^2 u(x,\omega) = q(x,\omega), \tag{2}$$

in this equation, $u(x,\omega)$ and $q(x,\omega)$ are the products of the Fourier transform of $U(x,t)$ and $Q(x,t)$, respectively and ω is the angular frequency. It is noticeable, the following property of the Fourier transform is utilized in derivation of Eq. (2)

$$\mathcal{F}[f^{(n)}(t)] = (iw)^n \mathcal{F}[f(t)],$$

where $i = \sqrt{-1}$ is the imaginary unit. Dividing both sides of Eq. (2) by EI gives

$$u''''(x,\omega) - \frac{m\omega^2}{\mathrm{EI}}u(x,\omega) = \frac{q(x,\omega)}{\mathrm{EI}}.$$

This equation can be rewritten as

$$u'''' - \lambda^4 u = \phi(x), \tag{3}$$

where

$$\phi(x) = \frac{q(x,\omega)}{\mathrm{EI}}; \quad \lambda^4 = \frac{m\omega^2}{\mathrm{EI}},$$

in which λ is called the frequency parameter of the beam. It is informative to mention that Eq. (3) is the governing equation of the Euler–Bernoulli beam in the frequency domain. In the next section, we attempt to solve the aforesaid differential equation, utilizing the Green's function method.

3. Green's function

In this section, the Green's function of the problem is to be found. Herein, the left-hand side of equation Eq. (3) is denoted by $\mathcal{L}u(x)$. Hence, this equation takes the form of