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Non-axisymmetric consolidation of poroelastic multilayered materials with anisotropic permeability and compressible constituents

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A B S T R A C T

This paper presents an analytical layer-element solution to non-axisymmetric consolidation of multilayered poroelastic materials with anisotropic permeability and compressible constituents. By applying Fourier expansions, Hankel transforms and Laplace transforms to the state variables involved in the governing equations of poroelasticity with respect to the circumferential, radial and time coordinates, respectively, the analytical layer-element (i.e. a symmetric stiffness matrix) is derived, which describes the relationship between the transformed generalized stresses and displacements at the surface ($z = 0$) and those at an arbitrary depth z, considering the corresponding boundary and continuity conditions at the layer interfaces, the global stiffness matrix of a multilayered system is assembled in the transformed domain. The actual solutions in the physical domain are acquired by applying numerical quadrature schemes for the inversion of the Laplace–Hankel transform. Finally, numerical calculation is presented to investigate the influence of layering and poroelastic material parameters on consolidation process.

1. Introduction

The study of the consolidation process of poroelastic materials in geomechanics has been a subject of great interest and an object of numerous investigations [1–12]. However, most of this research simplistically considered the effect of the properties of the porous material and its filled pore fluid on the consolidation process, since an isotropic poroelastic system with incompressible constituents and isotropic permeability was assumed and analyzed. As we know from practice, the typical deposit process of natural geomaterials may lead to an enormous difference in permeability between different directions, especially for the horizontal and vertical permeability, whose difference may be an order of magnitude or even more. Thus, considering the effect of anisotropic permeability on the consolidation process of poroelasticity has practical significance [13–19]. On the other hand, for several problems such as soil with high saturation degree, saturated porous rock encountered in geomechanics and energy resource explorations, the compressibility of constituents should not be ignored as well [20–26]. Reviewing the past achievements, few studies considering the anisotropy of permeability and the compressibility of constituents at the same time are conducted on the consolidation of poroelastic medium. Booker and Carter [27] analyzed the rate of consolidation with a point sink in an elastic half space by considering the anisotropy of permeability and the compressibility of pore fluid, Chen [28] used the state vector method to analyze the axisymmetric consolidation of a layered half

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space with anisotropic permeability and compressible pore fluid, however, both of them did not investigate the effect of the compressibility of the solid skeleton on the solid behavior. From the viewpoint of generality, a further extension of their solutions to cover all the aspects of the material properties mentioned above is still needed. At present, a relatively comprehensive investigation may refer to Singh et al. [29–30], who studied the plane strain and axisymmetric consolidation problems, in which the compressibility of constituents (both of the pore fluid and the solid skeleton) and anisotropy of the permeability were discussed.

The present paper aims to extend the studied problem by Singh et al. [29–30] to a more complex non-axisymmetric consolidation problem, in which the anisotropic permeability, compressibility of the constituents, and stratification of the material are all considered. To analyze multilayered structures, many theories or methods may be worth studying, such as the Zig-Zag theories for stratified plates and shells [31], finite layer method [7–9] and transfer matrix approach [10–12,25,26,28,32] for elastic and poroelastic media. In this paper, the analytical layer-element method [19] is utilized for its efficiency and stability in numerical calculation. In order to derive the solution for the studied problem, a new derivation method is employed to solve the basic equations of Biot's consolidation with anisotropic permeability and compressible constituents, which can greatly simplify the solution derivation. With the help of Fourier expansion, Hankel transforms and Laplace transforms with respect to the circumferential, radial and time coordinates, the analytical layer-element of a single material layer is obtained in the transformed domain. Then, by employing appropriate boundary and continuity conditions at layer interfaces, the global stiffness matrix is assembled and solved. Once the solutions in the transformed domain are obtained, the actual solutions in the physical domain can be acquired by the inversion of the Laplace–Hankel transform. In order to check the accuracy of the numerical procedure, the presented solutions are compared with the results existing in Rajapakse and Senjuntichai [22]. After that, selected examples are given to analyze the effect of different material parameters on the response of consolidation.

2. Governing equations

With neglect of the body forces, the Navier equations in the cylindrical coordinate system can be written as follows [20]:

$$\nabla^2 u_r + \frac{1}{1-2v}\frac{\partial e}{\partial r} - \frac{1}{r}\left(\frac{2}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) - \frac{\alpha}{G}\frac{\partial p}{\partial r} = 0 \tag{1a}$$

$$\nabla^2 u_\theta + \frac{1}{1-2v}\frac{\partial e}{r\partial \theta} - \frac{1}{r^2}\left(u_\theta - 2\frac{\partial u_r}{\partial \theta}\right) - \frac{\alpha}{G}\frac{\partial p}{r\partial \theta} = 0 \tag{1b}$$

$$\nabla^2 u_z + \frac{1}{1-2v}\frac{\partial e}{\partial z} - \frac{\alpha}{G}\frac{\partial p}{\partial z} = 0 \tag{1c}$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator; $e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$ denotes the dilatation; u_r, u_θ and u_z stand for displacements in r, θ and z directions, respectively; $\alpha = \frac{3(v_u - v)}{B(1-2v)(1+v_u)}$ is the Biot–Willis coefficient, and B is the Skempton's pore pressure coefficient [33], here v is the drained Poisson's ration, v_u is the undrained Poisson's ratio whose range is $[v,1]$. The range of α and B are [0,1]. When $\alpha = 1$, the system is simplified to soil with incompressible solid constituents. If $\alpha = 1, v_u = 0.5$ or $B = 1, v_u = 0.5$, the system is simplified to soil with incompressible constituents; G is the shear modulus; p is the pore pressure (positive under compression).

Applying the operators $(\frac{\partial}{\partial r} + \frac{1}{r})$, $\frac{1}{r}\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial z}$ to the Eqs. (1a)–(1c), respectively, we can get the following equations:

$$\nabla^2\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)u_r + \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{1}{1-2v}e - \frac{\alpha}{G}p\right) - \frac{2}{r^3}\frac{\partial^2 u_r}{\partial \theta^2} - \left(\frac{2}{r^2}\frac{\partial^2}{\partial r\partial \theta} - \frac{2}{r^3}\frac{\partial}{\partial \theta}\right)u_\theta = 0 \tag{2a}$$

$$\nabla^2\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left(\frac{1}{1-2v}e - \frac{\alpha}{G}p\right) + \left(\frac{2}{r^2}\frac{\partial^2}{\partial r\partial \theta} - \frac{2}{r^3}\frac{\partial}{\partial \theta}\right)u_\theta = 0 \tag{2b}$$

$$\nabla^2\left(\frac{\partial u_z}{\partial z}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{1-2v}e - \frac{\alpha}{G}p\right) = 0 \tag{2c}$$

Constitutive equations can be taken in the form:

$$\sigma_{ij} = 2G\left(\varepsilon_{ij} + \frac{v}{1-2v}\delta_{ij}e\right) - \alpha\delta_{ij}p, \quad i,j = r,\theta,z \tag{3}$$

where σ_{ij} is the total stress component of the soil; ε_{ij} is the strain component, δ_{ij} is the Kronecker delta.

The mass conservation law is given by:

$$\frac{\partial}{\partial t}\left(\alpha e + \frac{p}{M}\right) = \frac{1}{\gamma_w}\left(k_r\left(\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2}\right) + k_z\frac{\partial^2 p}{\partial z^2}\right) \tag{4}$$