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Higher order finite difference method for the reaction and anomalous-diffusion equation ☆,☆☆

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ABSTRACT

In this paper, our aim is to study the high order finite difference method for the reaction and anomalous-diffusion equation. According to the equivalence of the Riemann–Liouville and Grünwald–Letnikov derivatives under the suitable smooth condition, a second-order difference approximation for the Riemann–Liouville fractional derivative is derived. A fourth-order compact difference approximation for second-order derivative in spatial is used. We analyze the solvability, conditional stability and convergence of the proposed scheme by using the Fourier method. Then we obtain that the convergence order is $O(\tau^2 + h^4)$, where τ is the temporal step length and h is the spatial step length. Finally, numerical experiments are presented to show that the numerical results are in good agreement with the theoretical analysis.

1. Introduction

Fractional differential equations have attracted increasing interest due to its playing a significant role in neurons, control, electromagnetism, biophysics, physics, regular variation in thermodynamics, mathematical, mechanics, signal and image processing, blood flow phenomena, etc. [1–5].

It is well known that it is difficult to find the analytical solutions of the fractional differential equations. Therefore, seeking numerical methods is an important task in the studies of fractional differential equations. In recent years, there have existed various numerical methods for fractional differential equations, for instance, finite difference method [6–14], finite element method [15–17], and so on.

In this paper, we numerically study the following reaction and anomalous-diffusion equation [18,19]:

$$\frac{\partial u(x,t)}{\partial t} = {}_{RL}D_{0,t}^{1-\alpha}\left(K_\alpha \frac{\partial^2 u(x,t)}{\partial x^2} - C_\alpha u(x,t)\right) + f(x,t), \quad \alpha \in (0,1),\ 0 < t \leqslant T,\ 0 < x < L, \tag{1}$$

subject to the initial, boundary value conditions

$$u(x,0) = 0, \quad 0 \leqslant x \leqslant L,$$

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$$u(0,t)=\varphi_1(t),\quad 0\leqslant t\leqslant T,$$

$$u(L,t)=\varphi_2(t),\quad 0\leqslant t\leqslant T,$$

where ${}_{RL}D_{0,t}^{1-\alpha}$ denotes the Riemann–Liouville derivative of order $1-\alpha$ defined by Podlubny [4]

$${}_{RL}D_{0,t}^{1-\alpha}u(x,t)=\frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t\frac{u(x,s)}{(t-s)^{1-\alpha}}ds,$$

K_α and C_α are two positive constants, $f(x,t)$, $\varphi_1(t)$ and $\varphi_2(t)$ are sufficiently smooth functions.

Upto now, some numerical methods are available for the reaction and anomalous-diffusion equation with the case $C_\alpha=0$. For example, Yuste and Acedo [20] proposed an explicit finite difference method, where the order of convergence was $O(\tau+h^2)$. In [21], Yuste proposed the weighted average finite difference method, where for different weighted parameter λ, he got different convergence order. Chen et al. [22] presented an implicit scheme, in which the order of convergence was $O(\tau+h^2)$. Cui [23] obtained an unconditionally stable finite difference scheme, in which the order of convergence was $O(\tau+h^4)$. For the above Eq. (1), Chen et al. [24] proposed the implicit and explicit finite difference schemes, and got the convergence with the order $O(\tau+h^2)$. Very recently, Ding and Li [25] constructed a class of numerical methods and obtained different convergence orders by choosing different spline parameters. From the references available, it seems easy to increase the accuracy in spatial direction but difficult to increase the accuracy in time direction. In the present paper, a higher order approach in time direction for the numerical treatment of Eq. (1) is derived.

The outline of the rest of this paper is organized as follows. In Section 2, a numerical method for solving the reaction and anomalous-diffusion equation is proposed. The solvability, stability and convergence are analyzed in Sections 3 and 4, respectively. In Section 5, numerical experiments are carried out to support the theoretical analysis. And the conclusion is included in the last section.

2. Numerical method

In this section, we present an effective numerical method to simulate the solution of the reaction and anomalous-diffusion (1).

To establish the numerical scheme for the above Eq. (1), we let $x_i=ih\ (i=0,1,\ldots,M)$ and $t_k=k\tau\ (k=0,1,\ldots,N)$, where $h=\frac{L}{M}$ and $\tau=\frac{T}{N}$ are the uniform spatial and temporal step sizes respectively, and $M,\ N$ are two positive integers.

Firstly, using the Taylor series expansion at point (x_i,t_k), one gets

$$u(x_i,t_{k+1})=u(x_i,t_k)+\tau\frac{\partial u(x_i,t_k)}{\partial t}+\frac{\tau^2}{2}\frac{\partial^2u(x_i,t_k)}{\partial t^2}+\cdots=\left(\mathcal{I}+\tau\frac{\partial}{\partial t}+\frac{\tau^2}{2}\frac{\partial^2}{\partial t^2}+\ldots\right)u(x_i,t_k)=\exp\left(\tau\frac{\partial}{\partial t}\right)u(x_i,t_k),\tag{2}$$

where $\mathcal{I}$ is the identity operator.

For $\exp\left(\tau\frac{\partial}{\partial t}\right)$, we can use the following $(1,1)$ Padé approximation

$$\left\|\exp\left(\tau\frac{\partial}{\partial t}\right)\right\|=\left\|\left(2\mathcal{I}+\tau\frac{\partial}{\partial t}\right)\left(2\mathcal{I}-\tau\frac{\partial}{\partial t}\right)^{-1}\right\|+\mathcal{O}(\tau^3).\tag{3}$$

Applying the Taylor series expansion at point (x_i,t_k) (or using (2)), one can obtain

$$\frac{1}{2}\left[\frac{\partial u(x_i,t_k)}{\partial t}+\frac{\partial u(x_i,t_{k+1})}{\partial t}\right]=\frac{1}{\tau}[u(x_i,t_{k+1})-u(x_i,t_k)]+\mathcal{O}(\tau^2),$$

which can be rewritten as the following compact form

$$\left(\mathcal{I}+\frac{1}{2}\Delta_t\right)\frac{\partial u(x_i,t_k)}{\partial t}=\frac{1}{\tau}\Delta_tu(x_i,t_k)+\mathcal{O}(\tau^2),\tag{4}$$

where Δ_t denote forward difference operator with respect to t, defined by $\Delta_tu(x_i,t_k)=u(x_i,t_{k+1})-u(x_i,t_k)$.

Secondly, we focus on an approximation for the Riemann–Liouville derivative. Due to the equivalence between Riemann–Liouville derivative and Grünwald–Letnikov derivative under smooth condition, we usually approximate the Riemann–Liouville derivative by using the following Grünwald–Letnikov formula if the homogeneous condition satisfies:

$${}_{RL}D_{0,t}^{1-\alpha}u(x,t)=\frac{1}{\tau^{1-\alpha}}\sum_{j=0}^{\left[\frac{t}{\tau}\right]}\varpi_{1,j}^{(1-\alpha)}u(x,t-j\tau)+\mathcal{O}(\tau),\tag{5}$$

where $\left[\frac{t}{\tau}\right]$ denotes the integer part of $\frac{t}{\tau}$, the coefficients are

$$\varpi_{1,j}^{(1-\alpha)}=(-1)^j\binom{1-\alpha}{j}=(-1)^j\frac{\Gamma(2-\alpha)}{\Gamma(j+1)\Gamma(2-\alpha-j)},\quad j=0,1,\ldots$$