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Finite volume and finite element methods for solving a one-dimensional space-fractional Boussinesq equation ☆

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ABSTRACT

In this paper, a new one-dimensional space-fractional Boussinesq equation is proposed. Two novel numerical methods with a nonlocal operator (using nodal basis functions) for the space-fractional Boussinesq equation are derived. These methods are based on the finite volume and finite element methods, respectively. Finally, some numerical results using fractional Boussinesq equation with the maximally positive skewness and the maximally negative skewness are given to demonstrate the strong potential of these approaches. The novel simulation techniques provide excellent tools for practical problems. These new numerical models can be extended to two- and three-dimensional fractional space-fractional Boussinesq equations in future research where we plan to apply these new numerical models for simulating the tidal water table fluctuations in a coastal aquifer.

1. Introduction

In coastal aquifers, the groundwater table is greatly fluctuated by oceanic tides and wave actions [1]. The one-dimensional Boussinesq equation is often used to help understand and analyze coastal aquifer behavior, particularly the prediction of the tidal water table fluctuations [2,3].

The principle behind Boussinesq formulations is to incorporate the effects of non-hydrostatic pressure, while eliminating the vertical coordinate, thus significantly reducing the computational effort relative to a full three-dimensional solution. This principle was initially introduced by Boussinesq (1872), who derived new governing equations under the assumption that the magnitude of the vertical velocity increases polynomially from the bottom to the free surface which inevitably leads to some form of depth limitation in the accuracy of the embedded dispersive and nonlinear properties [4]. Hence, Boussinesq-type equations are conventionally associated with relatively shallow water.

Due to the increasing error in the modeled linear dispersion relation with increasing water depth, the standard Boussinesq equations are limited to relatively shallow water. Recently, efforts were made by a number of investigators to derive alternative forms of Boussinesq equations that can be applied in deeper water regions. In order to make the Boussinesq equations applicable in deeper water, many researchers have suggested ways to extend the validity range of the equations. The extended Boussinesq equations normally have adjustable polynomial approximations for the exact

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dispersion relationship, a major improvement from the standard ones. In order to make the Boussinesq equations applicable to porous media, many researchers have suggested ways to extend the validity range of the equations. The main assumption in using the Boussinesq equation is that ground water flow in a shallow aquifer can be described using the Dupuit–Forchheimer approximation, in which it is assumed that changes in hydraulic head with depth below the water table are negligible. The predictive ability of the solution was tested against field data from ten wells and ten piezometers on a microtidal sandy beach. Two model runs were carried out: in the first the effect of tides only was considered; in the second the setup due to waves was also considered. The piezometer data show that changes in hydraulic head with depth are negligible for much of the beach, suggesting that the use of the Boussinesq equation is appropriate [5].

Recently, fractional differential equations have attracted considerable interest because of their ability to model particle transport in heterogeneous porous media and complex phenomena [6]. Zhang et al. investigated the time and space nonlocalities underlying fractional-derivative models as a possible explanation for regional-scale anomalous dispersion with heavy tails. Properties of four fractional-order advectiondispersion equation (FADE) models were analyzed and compared systematically [7]. These models include five fractional advection–dispersion models (FADM), i.e., the immobile, mobile/immobile time FADM with a time Caputo fractional derivative $0 < \gamma < 1$; the space FADM with two sided Riemann–Liouville derivatives, the time–space FADM and the time fractional advection–diffusion-wave model with damping index $1 < \gamma < 2$. These equations can be used to simulate the regional-scale anomalous dispersion in heterogeneous porous media. Zhang et al. also investigated the space-fractional advection–dispersion equation (SFADE) with space dependent coefficients [8]. Some numerical methods for solving the space, time and time–space fractional models have been presented. Many researchers have proposed various numerical methods to solve space or time fractional partial differential equations during the past decade [9,10]. He proposed an approximate analytical solution for the fractional percolation equation via the variational iteration method [11]. As is well known, analytic solutions of most fractional differential equations cannot be obtained explicitly, so many authors resort to numerical solution strategies based on a rigour convergence and stability analysis. Meerschaert et al. (2004, 2006) presented finite difference methods to solve the one-dimensional space FADE and two-sided space-fractional differential equations [12,13]. Momani et al. developed an efficient algorithm based on the Adomian decomposition method for a time-fractional Navier–Stokes equation in the form of a rapidly convergent series with easily computable components [14], however, they did not give its theoretical analysis. Liu et al. proposed an approximation of the *Lévy − Feller* advection–dispersion process by a random walk and finite difference method, and discussed its stability and convergence [15]. Zhuang et al. proposed a new implicit numerical method and two solution techniques to improve its convergence order for solving an anomalous subdiffusion equation [16]. Liu et al. presented the numerical simulation of three-dimensional seepage flow with fractional derivatives in porous media [17]. Currently, the finite difference method (FDM), is a dominant numerical method used for solving fractional partial differential equations. Some researchers considered the finite element method (FEM) [18] or spectral method to handle the space derivative. Fix et al. [19] and Ervin et al. [20] developed finite element methods for the one-dimensional partial differential equations with constant coefficients on the fractional derivative terms. Roop [21] investigated the computational aspects for the Galerkin approximation using continuous piecewise polynomial basis functions on a regular triangulation of a bounded domain in $\mathbb{R}^2$. Li et al. [22] also studied the Galerkin finite element method of the time–space fractional order nonlinear subdiffusion and superdiffusion equations for a one-dimensional case. Zhao and Li [23,24] propose fully discrete Galerkin finite element method to solve the generalized nonlinear fractional Fokker–Planck equation and time–space fractional order telegraph equation. Bueno-Orovio et al. [25] introduced Fourier spectral methods for the fractional in space reaction–diffusion equations. Zhang et al. [26] presented a finite volume approach to solve an advection–dispersion equation with a constant dispersion coefficient. Hejazi et al. [27] proposed a finite volume method to solve the time–space two-sided fractional advection–dispersion equation on a one-dimensional domain. The spatial discretisation employs fractionally-shifted *Grünwald* formulas to discretise the Riemann–Liouville fractional derivatives at control volume faces in terms of function values at the nodes. These finite volume methods are not extended to two- and three-dimensional problems in a natural manner. Chen et al. [28] propose the Kansa method to solve fractional diffusion equations. Fu et al. [29] develop a novel boundary meshless approach, Laplace transformed boundary particle method (LTBPM), for numerical modeling of time fractional diffusion equations. However, effective numerical methods for solving the nonlinear fractional Boussinesq equations are still limited. This motivates us to develop computationally new numerical methods.

The space-fractional Boussinesq equation is nonlocal by definition, since it describes the spread of the height of a water table over large distances via a convolutional fractional derivative.

The main aim of this paper is to develop a new space-fractional Boussinesq equation (SFBE) in heterogenous porous media and propose a finite volume method and a finite element method for simulating space-fractional Boussinesq equations. These new numerical models will be used to simulate the tidal water table fluctuations in a coastal aquifer.

The paper is organized as follows. A new space-fractional Boussinesq equation is proposed in Section 2. Some properties of the nodal based functions and their fractional derivatives are introduced in Section 3. A finite volume method and a finite element method for solving the SFBE are proposed in Sections 4 and 5, respectively. Some numerical results are shown in Section 6 and it is demonstrated that these techniques can be used to simulate practical problems even when a complex transition zone is involved. Finally, the conclusions are given.