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A new fractional finite volume method for solving the fractional diffusion equation ☆

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ABSTRACT

The inherent heterogeneities of many geophysical systems often gives rise to fast and slow pathways to water and chemical movement. One approach to model solute transport through such media is by fractional diffusion equations with a space–time dependent variable coefficient. In this paper, a two-sided space fractional diffusion model with a space–time dependent variable coefficient and a nonlinear source term subject to zero Dirichlet boundary conditions is considered.

Some finite volume methods to solve a fractional differential equation with a constant dispersion coefficient have been proposed. The spatial discretisation employs fractionally-shifted *Grünwald* formulas to discretise the Riemann–Liouville fractional derivatives at control volume faces in terms of function values at the nodes. However, these finite volume methods have not been extended to two-dimensional and three-dimensional problems in a natural manner. In this paper, a new weighted fractional finite volume method with a nonlocal operator (using nodal basis functions) for solving this two-sided space fractional diffusion equation is proposed. Some numerical results for the Crank–Nicholson fractional finite volume method are given to show the stability, consistency and convergence of our computational approach. This novel simulation technique provides excellent tools for practical problems even when a complex transition zone is involved. This technique can be extend to two-dimensional and three-dimensional problems with complex regions.

1. Introduction

Diffusion equations (DE) and advection–dispersion equations (ADE) are well studied for a variety of potential types, and respective results have found wide application. In many studies of diffusion processes where the diffusion takes place in a highly non-homogeneous medium, the traditional ADE/DE may not be adequate (see [1–3]). In particular, the corresponding probability density of the concentration field may have a heavier tail than the Gaussian density, and its correlation function may decay to zero at a much slower rate than the usual exponential rate of Markov diffusion, resulting in long-range dependence. This phenomenon is known as anomalous diffusion (see [4]). Meerschaert and Sikorskii (see [5]) also studied

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stochastic models for fractional calculus. Leonenko et al. (see [6]) considered fractional pearson diffusions. Kochubei (see [7]) discussed fractional-parabolic systems. Fractional derivatives play a key role in modelling particle transport in anomalous diffusion including the space fractional diffusion equation (FDE)/space fractional advection–dispersion equation (FADE) describing *Lévy* flights (see [3]), the time FDE/FADE depicting traps, and the time–space FDE/FADE characterizing the competition between *Lévy* flights and traps (see [8]). A class of FDE/FADE has been successfully used to describe nonlocal dependence on either time and/or space, to explain the development of anomalous dispersion. These equations can be used to simulate regional-scale anomalous dispersion with heavy tails, for example, the solute transport in watershed catchments and rivers (see [9,10]). The spatial complexity of the environment imposes geometric constraints on transport processes on all length scales, that can be interpreted as temporal correlations on all time scale. Nonhomogeneities of the medium may fundamentally alter the laws of Markov diffusion, leading to long range fluxes, and non-Gaussian, heavy tailed profiles, and these motions may no longer obey Fick's Law. It is in this setting that fractional models can offer insights that traditional approaches do not offer.

In order to capture the observed local variation of transport speed, an extension of the homogeneous space FDE/FADE to a FDE/FADE with space-dependent and time-dependent coefficients has been suggested (see [9]). In this paper, we consider the following space fractional diffusion equation with a variable diffusion coefficient $K(x,t)$ on $(x,t) \in [(a,b) \times (0,T)]$:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left\{K(x,t)\left[\beta\frac{\partial^{\alpha}}{\partial x^{\alpha}} + (1-\beta)\frac{\partial^{\alpha}}{\partial(-x)^{\alpha}}\right]\right\}u(x,t) + f(u,x,t), \tag{1}$$

$$u(x,0) = \psi(x), \quad a \leqslant x \leqslant b, \tag{2}$$

$$u(x,t) = 0, \quad \text{for } x \in (-\infty, a] \text{ or } [b, +\infty),\ 0 \leqslant t \leqslant T,. \tag{3}$$

where $u(x,t)$ is the concentration; $K(x,t)$ $(0 < \underline{K} \leqslant K(x,t) \leqslant \overline{K})$ denotes the diffusion coefficient; $0 \leqslant \beta \leqslant 1$ indicates the relative weight of forward versus backward transition probabilities and $f(u,x,t)$ is a source (or absorbent) term that satisfies the Lipschitz condition.

Remark 1. Let X be a Banach space with associated norm $\|u\|$. We say that $f: X \to X$ is globally Lipschitz continuous if for some $L > 0$, we have $\|f(u) - f(v)\| \leqslant L\|u - v\|$ for all $u, v \in X$, and is locally Lipschitz continuous, if the latter holds for $\|u\|, \|v\| \leqslant M$ with $L = L(M)$ for any $M > 0$ [11]. For many problems of practical interest, the function f will not be globally Lipschitz on X. For example, in applications to population biology the most common model is the Kolmogorov–Fisher equation with $f(u) = ru(1 - u/K)$ and here f is not globally Lipschitz. Baeumer et al. [11] show how to solve nonlinear reaction–diffusion equations of type (1) by an operator splitting method when the abstract function f is only locally Lipschitz (see [11]).

The operators $\frac{\partial^{\alpha}}{\partial x^{\alpha}}$ and $\frac{\partial^{\alpha}}{\partial(-x)^{\alpha}}$ are defined as the left-sided and right-sided fractional Riemann–Liouville integral operators with $0 < \alpha < 1$:

$$\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x}\int_{-\infty}^{x}(x-\xi)^{-\alpha}u(\xi,t)d\xi, \tag{4}$$

$$\frac{\partial^{\alpha} u(x,t)}{\partial(-x)^{\alpha}} = -\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x}\int_{x}^{+\infty}(\xi-x)^{-\alpha}u(\xi,t)d\xi, \tag{5}$$

where $\Gamma(\cdot)$ is the Gamma function.

Hence, we incorporate zero Dirichlet boundary conditions in (3) and so the integrals in (4) and (5) are then defined the interval $[a,b]$.

There has been significant interest in developing numerical methods for solving equations of the form (1). Liu et al. [3] considered the numerical solution of the space fractional Fokker–Planck equation. They transformed the space fractional Fokker–Planck equation into a system of ordinary differential equations that was then solved using backward differentiation formulas (fractional method of lines). Meerschaert and Tadjeran [12] presented a finite difference method to solve the one-dimensional fractional advection–dispersion equations with a Riemann–Liouville fractional derivative on a finite domain. Meerschaert and Tadjeran [13] proposed shifted *Grüwald* formula to solve the two-sided space-fractional partial differential equations. Liu et al. [14] also considered a space–time fractional advection–diffusion equation with Caputo time fractional derivative and Riemann–Liouville space fractional derivatives. They proposed an implicit difference method and an explicit difference method, and discussed the stability and convergence of these numerical methods. Shen et al. presented explicit and implicit difference approximations for the Riesz fractional advection–dispersion equation and the space–time Riesz–Caputo fractional advection–diffusion equation in [15,16], respectively. Shen et al. [17] also proposed a second-order accurate finite difference approximation for the Riesz space fractional advection–dispersion equation. Zhuang et al. [18] developed a new implicit numerical method for the anomalous subdiffusion equation, which involves one fractional temporal derivative in the diffusion term. Liu et al. [19] considered both numerical and analytical