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A simple formula to find the closest consistent matrix to a reciprocal matrix

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ABSTRACT

Achieving consistency in pair-wise comparisons between decision elements given by experts or stakeholders is of paramount importance in decision-making based on the AHP methodology. Several alternatives to improve consistency have been proposed in the literature. The linearization method (Benítez et al., 2011 [10]), derives a consistent matrix based on an original matrix of comparisons through a suitable orthogonal projection expressed in terms of a Fourier-like expansion. We propose a formula that provides in a very simple manner the consistent matrix closest to a reciprocal (inconsistent) matrix. In addition, this formula is computationally efficient since it only uses sums to perform the calculations. A corollary of the main result shows that the normalized vector of the vector, whose components are the geometric means of the rows of a comparison matrix, gives the priority vector only for consistent matrices.

1. Introduction

The AHP (Analytic Hierarchy Process) [1,2] is designed for multi-objective, multi-criteria, and multi-actor decisions, with and without certainty, for any number of alternatives. The AHP approach mainly consists of three stages, construction of the hierarchy of problem ingredients, namely, objective, criteria and alternatives, calculation of the priorities of the elements, and aggregation of results to produce the final decision. Interactions between the elements are considered when building the structure of the problem. The elements are evaluated using pairwise comparisons, by asking experts or stakeholders involved in the decision-making problem about how much importance a criterion has when compared with another criterion with respect to the interests or preferences of respondents. The candidate alternatives are also evaluated by pairwise comparisons with respect to what is the higher degree of satisfaction for each criterion.

Both kinds of related values can be determined by using various scales, in particular the scale of 1–9 to represent equal importance to extreme importance [1]. Performing such a comparison yields an $n \times n$ matrix $A = (a_{ij})$, whose (positive) entries must adhere to two important properties, namely, $a_{ii} = 1$ (homogeneity) and $a_{ji} = 1/a_{ij}$ (reciprocity), $i, j = 1, \ldots, n$. The problem for matrix A becomes one of producing for the n elements, $E_1, \ldots, E_n$ (criteria or alternatives) under comparison, a set of numerical values $w_1, \ldots, w_n$ that reflect the priorities among the compared elements according to the emitted judgments. If all the judgments are completely consistent, the relations between weights w_i and judgments a_{ij} are simply

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given by $w_i/w_j = a_{ij}$ (for $i,j = 1, 2, \ldots, n$) and the matrix A is said to be consistent. The following theorem provides equivalent conditions for a matrix A to be consistent.

Firstly, we provide some notation. $M_{n,m}$ will hereinafter denote the set of $n \times m$ real matrices, and $M^+_{n,m}$ will denote the subset of $M_{n,m}$ composed of positive matrices. It will be assumed that the elements of $\mathbb{R}^n$ are column vectors, i.e., $\mathbb{R}^n$ is identified with $M_{n,1}$. For a given $A \in M_{n,m}$, let us write $[A]_{ij}$ the (i,j) entry of the matrix A. The superscript T denotes the matrix transposition. The mapping $J : M^+_{n,m} \to M^+_{n,m}$ defined by $[J(A)]_{ij} = 1/[A]_{ij}$ will play an important role in the sequel.

Theorem 1 [3, Theorem 1]. *Let $A = (a_{ij}) \in M^+_{n,n}$. The following statements are equivalent.*

(i) *There exists $\mathbf{x} \in M^+_{n,1}$ such that $A = J(\mathbf{x})\mathbf{x}^T$.*
(ii) *There exists $\mathbf{w} = [w_1 \cdots w_n]^T \in M^+_{n,1}$ such that $a_{ij} = w_i/w_j$ for all $i,j \in \{1, \ldots, n\}$.*
(iii) *$a_{ij}a_{ji} = 1$ and $a_{ij}a_{jk} = a_{ik}$ hold for all $i,j,k \in \{1, \ldots, n\}$.*

For a consistent matrix, the leading eigenvalue and the principal (Perron) eigenvector of a comparison matrix provide information to deal with complex decisions, the normalized Perron eigenvector giving the sought priority vector [2]. It is also well known that any consistent matrix has rank one [3], and as a consequence, any of its normalized rows and, in particular, the normalized vector of the geometric means of the rows, also provides the priority vector. Taking into account the (natural lack of) consistency of human thinking, some degree of inconsistency is expected and, as a result, in general A is not consistent. As shown in [4] the eigenvector is necessary for obtaining priorities. The hypothesis that the estimates of these values are small perturbations of the "right" values guarantees a small perturbation of the eigenvalues (see, e.g. [5]). For non-consistent matrices, the problem to solve is the eigenvalue problem $A\mathbf{w} = \lambda_{\max}\mathbf{w}$, where $\lambda_{\max}$ is the unique largest eigenvalue of A that gives the Perron eigenvector as an estimate of the priority vector. As a measurement of inconsistency, Saaty proposed using the consistency index $\mathrm{CI} = (\lambda_{\max} - n)/(n-1)$ and the consistency ratio $\mathrm{CR} = \mathrm{CI}/\mathrm{RI}$, where RI is the so-called average consistency index [2]. If $\mathrm{CR} < 0.1$, the estimate is accepted; otherwise, a new comparison matrix is solicited until $\mathrm{CR} < 0.1$.

Achieving consistency in AHP has become an important issue and different methods have been proposed in the literature [1–3,6–14]. In this paper, we focus on the linearization process [10] not as method to directly obtain the priority vector, but as a method that provides a closed form for achieving complete consistency. Here we use the word closed in contrast with methods relying on optimisation, which is non-linear for this problem, and are iterative by nature. Achieving complete consistency is a feature that may be suitably used for specific purposes.

In Section 2 we provide a short review of the linearization method. In Section 3 we develop a simple formula to obtain the consistent matrix that is closest to a given comparison matrix and its associated priority vector. This formula involves just sums, a very important computational feature. As a consequence, we show that the row geometric mean method (RGMM) gives the priority vector only for completely consistent matrices. Finally, a section devoted to discussion and conclusions closes the paper.

2. Short review of the linearization method

Let us recall that a reciprocal matrix $A \in M^+_{n,n}$ verifies the condition $A_{ij} = 1/A_{ji}$ for $1 \leqslant i,j \leqslant n$, whereas a consistent matrix $A \in M^+_{n,n}$ also satisfies $A_{ij}A_{jk} = A_{ik}$ for $1 \leqslant i,j,k \leqslant n$.

As we have mentioned, an important problem in AHP theory is the following: find the closest consistent matrix to a given reciprocal matrix $A \in M^+_{n,n}$. To this end, in [10] the mappings were introduced

$$L : M^+_{n,n} \to M_{n,n}, \quad [L(A)]_{ij} = \log[A_{ij}]$$

and

$$E : M_{n,n} \to M^+_{n,n}, \quad [E(A)]_{ij} = \exp[A_{ij}].$$

Each of these mappings is, evidently, one the inverse of the other. Obviously, for a given $A \in M^+_{n,n}$ we have

$$A \text{ is reciprocal} \iff L(A) \text{ is skew-Hermitian}.$$

Furthermore, in [10, Theorem 2.2] it was proven that

$$\mathcal{L}_n = \{L(A) : A \in M^+_{n,n} \text{ and } A \text{ is consistent}\} \text{ is a linear subspace}. \tag{1}$$

The aforementioned approximation problem was solved by means of a linearization technique [10].

We need some notation to state this solution: We consider all vectors of $\mathbb{R}^n$ as column vectors, by $\mathbf{1}_n$ we denote the vector of $\mathbb{R}^n$ given by $\mathbf{1}_n^T = (1, \ldots, 1)$, the trace operator is denoted by $tr(\cdot)$, i.e., for a square matrix $A \in M_{n,n}$, $tr(A) = [A]_{1,1} + \cdots + [A]_{n,n}$, and finally, ϕ_n denotes the linear mapping defined by

$$\phi_n(\mathbf{x}) = \mathbf{x}\mathbf{1}_n^T - \mathbf{1}_n\mathbf{x}^T, \quad \phi_n : \mathbb{R}^n \to M_{n,n}. \tag{2}$$

The mathematical tool to solve the approximation problem is given by the following result.