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Optimizing fuzzy *p*-hub center problem with generalized value-at-risk criterion

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ABSTRACT

In this study, we reduce the uncertainty embedded in secondary possibility distribution of a type-2 fuzzy variable by fuzzy integral, and apply the proposed reduction method to *p*-hub center problem, which is a nonlinear optimization problem due to the existence of integer decision variables. In order to optimize *p*-hub center problem, this paper develops a robust optimization method to describe travel times by employing parametric possibility distributions. We first derive the parametric possibility distributions of reduced fuzzy variables. After that, we apply the reduction methods to *p*-hub center problem and develop a new generalized value-at-risk (VaR) *p*-hub center problem, in which the travel times are characterized by parametric possibility distributions. Under mild assumptions, we turn the original fuzzy *p*-hub center problem into its equivalent parametric mixed-integer programming problems. So, we can solve the equivalent parametric mixed-integer programming problems by general-purpose optimization software. Finally, some numerical experiments are performed to demonstrate the new modeling idea and the efficiency of the proposed solution methods.

1. Introduction

The study of *p*-hub center problem was firstly introduced by Campbell [1], in which a quadratic programming model was formulated. Kara and Tansel [2] provided several linear formulations for the single allocation *p*-hub center problem. Ernst et al. [3] proposed a mixed-integer linear programming for the multiple allocation *p*-hub center problem based on the concept radius of hubs. For comprehensive surveys on the deterministic *p*-hub center problems, the interested reader may refer to Alumur and Kara [4] and Campbell et al. [5].

In this paper, we concentrate on *p*-hub center problem under uncertainty, which is an active research area in the literature. The significance of uncertainty has motivated some researchers to address hub location problems with random hub nodes [6], random demands [7], random costs [8] and random travel times [9,10]. In the meantime, some new methods have also been developed to model hub location problems under possibilistic uncertainty. For example, Bashiri et al. [11] used fuzzy VIKOR to model a hub location problem, in which the location of hub facilities is determined by both qualitative and quantitative parameters. Taghipourian et al. [12] presented a fuzzy dynamic model, in which a group of facilities is considered as virtual hubs for backup in case the main hubs fail to operate. Yang et al. [13] investigated hub location problems based on the credibility criterion, and developed local search based hybrid algorithms to solve their optimization problem.

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However, for hub location problem, the exact possibility distributions of fuzzy travel times are sometimes difficult determined. It is necessary to find ways to evade the exact evaluation of possibility distributions of uncertain travel times. Type-2 fuzzy vector [14,] is a natural extension of type-1 fuzzy vector, it provides an appropriate representation of uncertain parameters in certain applications [15,16]. During the applications of type-2 fuzzy theory in practice, several type-reduction methods have been developed in the literature [17–22]. Motivated by the above mentioned research, the purpose of this paper is to develop a robust parametric method for optimizing p-hub center problem, in which the uncertain travel times are characterized by type-2 fuzzy variables. We employ Choquet integrals [23] to reduce the uncertainty embedded in the secondary possibility distributions of uncertain travel times, and develop a novel fuzzy p-hub center problem. The Choquet integral was first introduced in [24], and its properties were also documented in [25,26]. When the travel times are mutually independent type-2 fuzzy parameters, we transform the established p-hub center problem to its equivalent parametric mixed-integer programming problems. In addition, the equivalent mixed-integer programming problems can be solved by general-purpose optimization softwares. At the end of this paper, we perform some numerical experiments to illustrate the new modeling ideas and the efficiency of the proposed solution methods.

The rest of this paper is organized as follows. In Section 2, we deal with the reduction methods for type-2 trapezoidal fuzzy variables, and deduce the parametric possibility distributions of reduced fuzzy variables. In Section 3, we develop a new modeling approach to p-hub center problem, and discuss the equivalent parametric programming problems. Section 4 performs some numerical experiments to illustrate the new modeling idea. Section 5 gives our conclusions.

2. Parametric possibility distributions of uncertain parameters

If a fuzzy variable ξ takes its values in the unit interval $[0,1]$, then it is called a regular fuzzy variable [14]. Suppose $\mu_\xi(t)$ is a generalized possibility distribution (not necessarily normalized) of regular fuzzy variable ξ. Then for any $t \in [0,1]$, the possibility, necessity and credibility of fuzzy event $\{\xi \leqslant t\}$ are calculated by the following formulas:

$$\mathrm{Pos}\{\xi \leqslant t\} = \sup_{0 \leqslant u \leqslant t} \mu_\xi(u), \tag{1}$$

$$\mathrm{Nec}\{\xi \leqslant t\} = \sup_{0 \leqslant u \leqslant 1} \mu_\xi(u) - \sup_{t < u \leqslant 1} \mu_\xi(u) \tag{2}$$

and

$$\mathrm{Cr}\{\xi \leqslant t\} = \frac{1}{2}\left(\sup_{0 \leqslant u \leqslant 1} \mu_\xi(u) + \sup_{0 \leqslant u \leqslant t} \mu_\xi(u) - \sup_{t < u \leqslant 1} \mu_\xi(u)\right). \tag{3}$$

If the possibility distribution $\mu_\xi(t)$ is normalized, i.e., $\sup_{0 \leqslant u \leqslant 1} \mu_\xi(u) = 1$, then the credibility defined above coincides with the concept defined in ([23]).

From the measure-theoretic interpretation of Choquet integral [24,26], it is usually regarded as the generalization of usual mathematical expectation. Therefore, motivated by the idea of Choquet integral, Liu and Liu [23] presented the following definition about the expected value of fuzzy variable.

Definition 1 [23]. Let ξ be a normalized fuzzy variable. The upper expected value, $E^*[\xi]$, of ξ is defined by

$$E^*[\xi] = \int_0^{+\infty} \mathrm{Pos}\{\xi \geqslant r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Nec}\{\xi \leqslant r\}\mathrm{d}r,$$

while the lower expected value, $E_*[\xi]$, of ξ is defined by

$$E_*[\xi] = \int_0^{+\infty} \mathrm{Nec}\{\xi \geqslant r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Pos}\{\xi \leqslant r\}\mathrm{d}r.$$

The expected value of ξ is defined as

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \geqslant r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \leqslant r\}\mathrm{d}r.$$

According to the definitions of the lower expected value, expected value, and upper expected value of fuzzy variable, for a regular triangular fuzzy variable $\xi = (r_1, r_2, r_3)$, we have the following computational formulas:

$$E_*[\xi] = \frac{r_1 + r_2}{2}, \quad E[\xi] = \frac{r_1 + 2r_2 + r_3}{4}, \quad E^*[\xi] = \frac{r_2 + r_3}{2}. \tag{4}$$

A type-2 fuzzy variable represents the uncertainty in terms of second possibility distribution. In this section, we adopt expectation method to reduce the uncertainty embedded in second possibility distribution. More precisely, let $\tilde{\xi}$ be a type-2 fuzzy variable. Then we define the lower expected value, expected value and upper expected value of regular fuzzy