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Double power laws, fractals and self-similarity

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ABSTRACT

Power law (PL) distributions have been largely reported in the modeling of distinct real phenomena and have been associated with fractal structures and self-similar systems. In this paper, we analyze real data that follows a PL and a double PL behavior and verify the relation between the PL coefficient and the capacity dimension of known fractals. It is to be proved a method that translates PLs coefficients into capacity dimension of fractals of any real data.

1. Introduction

Power law (PL) distributions, also known as heavy tail distributions, firstly appeared in the literature in the 19th century. In 1896 [1], Vilfredo Pareto applied a PL to model the distribution of individuals' incomes (this PL was later called Pareto law). He found that the relative number of individuals with an annual income larger than a certain value x was proportional to a power of x. From then on, studies of applications of PLs to real world phenomena have largely increased.

The most well known examples of PL distributions are the Pareto [1] and the Zipf [2,3] laws. The later is also known as rank-size rule. Let X be a non-negative discrete random variable following a PL distribution. Then, its complementary cumulative distribution function is of the form $F(x) = P(X \geqslant x) = \frac{C}{\alpha-1}x^{-(\alpha-1)}$, where $\alpha > 0$, $C > 0$. In the text, we will consider $\tilde{\alpha} = \alpha - 1$ and $\widetilde{C} = \frac{C}{\tilde{\alpha}}$. The probability function of a discrete random variable following Pareto distribution is given by:

$$P(X = x) = Cx^{-\alpha} \tag{1}$$

Zipf law is a special case of the Pareto law with exponent $\tilde{\alpha} = 1$.

Application of PL behavior in natural or human-made phenomena usually comes with a log–log plot, where the axes represent the size of an event and its frequency. The log–log plot is asymptotically a straight line with negative slope. We can consider, for example, a country, such as United States (US) or Portugal (PT), and order the cities by population. In the case of US, New York appears first, Los Angeles, second, and so on. Analogously for PT, Lisbon is the first, Oporto the second. Then, we can plot the logarithm of the rank on the y-axis and the logarithm of the city size on the x-axis. New York and Lisbon both have log rank ln 1, and Los Angeles and Oporto have log rank ln 2. The graphs are straight lines with negative slopes. The same thing happens for popularity. We all know that popularity is an extreme imbalanced phenomenon. Only a

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few people have access to the glare of the spotlight and fewer still manage that his/her name be recorded in history. Most of us go through life being known just to the people in our social circle. How is popularity related to city sizes? They have in common the power law behavior, that is, exhibiting, in a log–log scale plot, a straight line. In fact, power laws seem to reign in the study of phenomena where popularity of some kind is present. Another such example is popularity of web sites. It is found that the number of web pages that have k in-links follows a PL distribution [4]. The usefulness of PLs can be at the level of controlling the outcome of some phenomena. For example, in computer networks. Some typical behavior in these networks is individual agents acting in their own best interest, giving rise to a global power law. This can be changed by giving agents incentives to modify their conduct. Of course this type of strategy would not apply to earthquakes. Nevertheless, it could be used in an extremly important event, such as stock markets. Gopikrishnan et al. [5] described a PL behavior of stock market returns. Thus, comprehending PLs may be key to the understanding of stock market crashes [6], and many other important real life events. More examples are in wealth distribution and expenditure [7,8], city size distribution [9,3,10], number of articles' citations and scientific production [11,12], number of hits in webpages [4], number of victims in wars, terrorist attacks, and earthquakes [13–18], words' frequency [19,3] and occurence of personal names [20]. Interesting reviews on PL behavior and applications can be found in [21–23].

Most of the PLs application seen in the literature use a single PL to model the studied events. Nevertheless other PLs, such as double PLs also appear and are said to be a better fit in some cases [24–27].

Detecting or proving the existence of a PL behavior in natural or human-made systems can be a very difficult task. The modeling of PLs has been primarily theoretical. A different approach to find a more complete model should consider contributions from statistics, control theory, and economics [28].

Self-similar systems are characterized by being scale-free. This translates, in day-to-day English, in looking exactly the same, despite a closer or a more distant look. Fractals are ubiquitous in nature, appearing everywhere, from plant structures, body parts, such lungs, coastlines, mountain ranges, condensed-matter systems including polymers, composite materials, porous media, and other natural phenomena [29–32]. Self-similarity, self-invariance and fractal dimension are properties of fractals.

Mandelbrot [33], in 1982, wrote that.

Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.

Mandelbrot turned mathematicians, physicists, biologists, and other scientists', attentions to fractal patterns. The term "fractal dimension" [34] is frequently defined as the exponent D of the expression (2) given by:

$$n(\epsilon) = k\epsilon^{-D} \tag{2}$$

where n is the minimum number of open sets of diameter ϵ needed to cover the set and k is a constant that depends on the fractal size. This is also known as the capacity dimension of the fractal. There are other dimensions used to characterize fractals, being the Hausdorff's and Kolmogorov's dimensions, the ones that are more accurate, but also harder to use. Computing the fractal dimension of the length of a country's coastline is an extremely difficult task, that depends on the length of the ruler used in the measurements. The shorter the ruler the bigger the length, since a shorter ruler measures more accurately the sinuosity of bays and inlets. Doing a log–log plot of the length of the ruler s versus the measured length L of the coastline, a straight line, with slope between 1 and 2, is obtained. Mandelbrot computed this slope to be $1 - D$, so the analytical expression of the straight line is $\log L = (1 - D)\log s + b$. Last expression can be rewritten as $L = \tilde{b}s^{1-D}$, that is a PL, and $b = \log \tilde{b} \in \mathbf{R}$. This feature suggests an analogy between any phenomenon characterized by a PL distribution and the fractal dimension, having for the variables in the x- and y-axes the same role as ϵ and $n(\epsilon)$, and for the PL slope the fractal dimension D. One can say that $\tilde{\alpha}$ is analogous to D and, therefore, we can interpret the phenomenon in the perspective of geometric fractals. For example, if for a real case the PL reveals $\tilde{\alpha} \simeq 0.63$, one may say that the phenomenon has, in some way, similarities with the ternary Cantor fractal, whose fractal dimension is $D = \log 2/\log 3$ and that each object, or entity, in the phenomenon, is related with two smaller objects having $1/3$ the size each. Nevertheless, during the flow of recursion situations, the recursion scheme may vary. In that case the log–log plot changes and the different slopes reflect the distinct recursive laws. We will return to this subject with an example in Section 4.

PLs are extremely important in the study of systems that are self-similar or fractal-like over many orders of magnitude. PL behavior allows extrapolation and prediction over a wide range of scales. The study of scaling reveals itself as a powerful tool of simplifying systems complexity and of understanding the basic principles ruling those systems. Moreover, experimental data from self-similar systems cannot be described by any other statistical distributions, as Normal or exponential, since order in complex systems relies heavily on correlations between different levels of scale. For the sake of completeness, we remark that recently, work by Sornette [35] has driven attention to a new type of extreme events, labeled *dragon-kings*, that could not be predicted by the extrapolation of power law distributions. The detection of these *wild* events or outliers depends on the phenomena that is under observation, there is not a unique methodology to find them. Nevertheless, this is not the focus here in this work.

2. Fatal events

Understanding victimology patterns arising in fatal events, such as wars, terrorists attacks and tornadoes, is extremely important due to political, cultural, historical and geographical issues. Many researchers have attempted explanations in the last decades [13,14,36,15–17,37,18]. Nevertheless, understanding these patterns is still far from complete.