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# Utilization of characteristic polynomials in vibration analysis of non-uniform beams under a moving mass excitation

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ABSTRACT

Vibration of non-uniform beams with different boundary conditions subjected to a moving mass is investigated. The beam is modeled using Euler–Bernoulli beam theory. Applying the method of eigenfunction expansion, equation of motion has been transformed into a number of coupled linear time-varying ordinary differential equations. In non-uniform beams, the exact vibration functions do not exist and in order to solve these equations using eigenfunction expansion method, an adequate set of functions must be selected as the assumed vibration modes. A set of polynomial functions called as beam characteristic polynomials, which is constructed by considering beam boundary conditions, have been used along with the vibration functions of the equivalent uniform beam with similar boundary conditions, as the assumed vibration functions. Orthogonal polynomials which are generated by utilizing a Gram–Schmidt process are also used, and results of their application show no advantage over the set of simple non-orthogonal polynomials. In the numerical examples, both natural frequencies and forced vibration of three different non-uniform beams with different shapes and boundary conditions are scrutinized.



## 1. Introduction

The dynamic behavior of beams under moving loads (masses) is of great importance in several fields of engineering, for example, the design of road and railway bridges and the analysis of machining processes. Many engineers and scientists have contributed to the solution of the problem with their innovations, and still the dynamics of beams when subjected to moving loads is a subject that draws considerable attention of the researchers. A very helpful review of the literature can be found in Fryba [1] in which various analytical solutions for vibration problems of beam structures under moving loads have been investigated.

Transverse vibration of bridge structures acted upon by moving vehicles could be modeled by a beam structure traversed by moving loads. By including the inertial effects of the moving loads in the problem formulation, a more realistic formulation is achieved, which is known as a moving mass problem. Akin and Mofid [2] have solved the moving mass problem in Euler–Bernoulli beams, numerically, employing the method of separation of variables. Comprehensive studies on the dynamic stability of continuous systems under the effect of inertial forces and high velocities of moving loads have been stated by Verichev and Metrikine [3]. Dehestani et al. [4] have explored the critical influential speed of mass and showed the necessity of using the Coriolis acceleration, associated with the moving mass. Kiani et al. [5] have carried out a parametric study on transverse vibration of uniform beams with different beam theories via reproducing kernel particle method (RKPM). They

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reported the significance of moving mass weight and velocity as well as the assumed beam theory and boundary conditions. Nikkhoo et al. [6] have studied the effect of induced acceleration terms on the behavior of a uniform Euler–Bernoulli beam under the excitation of a moving mass using the eigenfunction expansion method. It has been shown that for the moving vehicles with relative large masses moving at high speeds, the importance of the inertial effects due to convective acceleration terms could not be ignored. Kiani et al. examined dynamics of multi-span viscoelastic thin [7] and thick [8] beams under a moving mass employing generalized moving least square method (GMLSM). Their results revealed the cruciality of the inertial effects for beams with more span numbers.

Dynamic behavior of uniform beams has been the subject of a majority of the conducted studies while in practice, non-uniform beams are widely used in an effort to achieve the optimum distribution of strength and weight rather than uniform beams and maybe to satisfy special architectural and functional requirements. In the case of beams with uniform inertia, cross-sectional area and mass, closed-form solutions exist, with simple formula for the natural frequencies and mode shapes. However, no such simple formula exists for beams in which inertia, cross-sectional area and mass vary along the length of the beam. Although methods are available to estimate the solution, determination of the dynamic properties of such beams has been and still is the subject of interest for several researchers.

Method of eigenfunction expansion has been widely used for dynamic analysis of structures. The main challenge in this method, which can be considered as a variant of the Galerkin method, is finding an adequate set of vibration functions that yields to the most accurate result with less computational effort. Like in other variational and weighted residuals methods, the only restrictions on the polynomial function series chosen is, that it satisfies the geometric boundary conditions, is complete, and does not inherently violate the natural boundary conditions [9]. When these conditions are met, solutions converge to the exact solution as more terms in the series are retained [10]. Trying to find natural frequencies of uniform rectangular plates with different boundary conditions, Bhat [9] used the Gram–Schmit process to generate orthogonal sets of polynomials in which higher terms of the series are recursively formed, based on the starting polynomials which are specific to the boundary conditions under consideration. A comprehensive study conducted by Chakraverty [11], has mainly devoted to the solution of the plates vibration problems using some variations of the method originally presented by Bhat [9].

As a general method, Chehil and Jategaonkar [12] and Jategaonkar and Chehil [13], have obtained a characteristic equation for determining the natural frequencies of non-uniform beams with different shapes of cross sections, by using mode shape functions of the equivalent uniform beam in a procedure based on the Galerkin method. Concerning the problem of beam structures under moving loads or masses, Zheng et al. [14] used modified beam vibration functions to analyze vibration of multi-span non-uniform beams with simple boundary conditions, excited by means of moving loads. The latter problem has been investigated via miscellaneous methods by other researchers [15,16,17]

However, it would be worthwhile to use an efficient method to cope with such a problem, which needs less computational effort and of course, be easy to manage. In this regard, the main contribution of this paper is to employ a simple and convenient set of polynomials in the eigenfunction expansion method, as assumed eigenfunctions or vibration modes, with the intention of analyzing the vibration of non-uniform beams with different boundary conditions. The proposed method has the advantages of simplicity, low computational effort and good precision.

## 2. Problem statement and formulation

### 2.1. Equation of motion

The governing differential equation for vibration of a non-uniform Euler–Bernoulli beam within the domain $X \in [0,1]$, could be written as:

$$m(x)\frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 v(x,t)}{\partial x^2}\right] = p(x,t), \tag{1}$$

where $v(x,t)$ denotes vertical deflection of point $x$ at time $t$. $m(x)$ and $EI(x)$ are, mass per unit length and flexural rigidity of the beam, respectively. For a one-dimensional beam problem, the position vector $x$ represents the $x$-axis with its origin coincided on the left support and $p(x,t)$ indicates the effect of external excitation on beam. In the case of a beam traversed by a mass $m_{mm}$ moving at the speed of $\dot{x}_{mm}$, it could be described by:

$$\begin{aligned} p(x,t) &= m_{mm}\left(g - \frac{\partial^2 v(x,t)}{\partial t^2}\right)_{x=\dot{x}_{mm}t}\delta(x-\dot{x}_{mm}t), \\ &= m_{mm}\left(g - \frac{\partial^2 v(x,t)}{\partial t^2} - 2\dot{x}_{mm}\frac{\partial^2 v(x,t)}{\partial x \partial t} - \dot{x}_{mm}^2\frac{\partial^2 v(x,t)}{\partial x^2}\right)_{x=\dot{x}_{mm}t}\delta(x-\dot{x}_{mm}t), \end{aligned} \tag{2}$$

where $g$ is the gravitational acceleration and $\delta(x-\dot{x}_{mm}t)$ is the Dirac delta function that represents the position of the moving mass with the speed of $\dot{x}_{mm}$ at time $t$. Moreover, the beam is assumed to be initially at rest and undeformed.

Using weighted residuals methods, deflection $v(x,t)$ can be considered as,