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Fractional multi-commodity flow problem: Duality and optimality conditions ☆

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ABSTRACT

This paper deals with multi-commodity flow problem with fractional objective function. The optimality conditions and the duality concepts of this problem are given. For this aim, the fractional linear programming formulation of this problem is considered and the weak duality, the strong direct duality and the weak complementary slackness theorems are proved applying the traditional duality theory of linear programming problems which is different from same results in Chadha and Chadha (2007) [1]. In addition, a strong (strict) complementary slackness theorem is derived which is firstly presented based on the best of our knowledge. These theorems are transformed in order to find the new reduced costs for fractional multi-commodity flow problem. These parameters can be used to construct some algorithms for considered multi-commodity flow problem in a direct manner. Throughout the paper, the boundedness of the primal feasible set is reduced to a weaker assumption about solvability of primal problem which is another contribution of this paper. Finally, a real world application of the fractional multi-commodity flow problem is presented.

1. Introduction

In many application contexts, several physical commodities, vehicles, or messages, each governed by their own network flow constraints, share the same network [2,3]. Multi-commodity flow problem is achieved in a lot of network design and transportation problems [4]. In this problem some commodities from special origins should be transmitted to some other destinations with a minimum total cost [2]. Solving this problem under integer restriction is NP-hard [2]. When multiple objective functions or uncertain ones are given, the solution process of multi-commodity problem is harder than traditional methods [5]. When only two objective functions are considered into account e.g., cost minimization and reliability maximization, one can optimize the ratio of these objective functions. The provided fractional programming problem can be solved applying some famous approaches. Besides, fractional programs are happened in a lot of practical problems [6,7] which can be pursued in other network analysis. For fractional multi-commodity problems to satisfy the demand for each commodity at each node without violating the constraints imposed by the supply–demand and capacity, one can consider the following

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maximum fractional multi-commodity flow model in which $G = (N, A)$ is a network with N and A as the sets of n nodes and m links:

$$maximize \quad f(x) = \frac{\sum_{k=1}^{K}\sum_{(i,j)\in A} c_{i,j}^k x_{i,j}^k + \alpha}{\sum_{k=1}^{K}\sum_{(i,j)\in A} d_{i,j}^k x_{i,j}^k + \beta}, \tag{1.1}$$

$$s.t. \quad \begin{cases} \sum_{j:(i,j)\in A} x_{i,j}^k - \sum_{j:(j,i)\in A} x_{j,i}^k = b_i^k, & \forall i \in N, \ \forall k = 1,\ldots,K, \\ \sum_{k=1}^{K} x_{i,j}^k \leqslant u_{i,j}, & \forall (i,j) \in A, \\ x_{i,j}^k \geqslant 0, & \forall (i,j) \in A, \ \forall k = 1,\ldots,K. \end{cases} \tag{1.2}$$

Here, $x_{i,j}^k$ is a nonnegative variable regarding to the amount of flow of kth commodity which streams through link (i,j), $u_{i,j}$ is the capacity of link (i,j) and K is the number of commodities. Moreover, for commodity k, $c_{i,j}^k$ and $d_{i,j}^k$ are the unit-reliability and the unit-cost of flow through link (i,j), respectively, and b_i^k is the supply or demand of node i, defined by positive or negative numbers. In this model α and β are two positive constants regarding two lower levels for the corresponding objective functions.

Applying vector and matrix notations, this model can be rewritten as the following maximum fractional linear programming model:

$$\begin{aligned} & maximize \quad f(x) = \tfrac{c^t x + \alpha}{d^t x + \beta} \\ & s.t. \\ & \qquad\qquad x \in S_1 = \{x : Ax \leqslant b, \ x \geqslant 0\}. \end{aligned} \tag{1.3}$$

Here A is a $(2.n.K + m)$ by $(m.K)$ matrix as follows:

$$A = \begin{array}{cc} \begin{array}{ccccc} k=1 & k=2 & \cdots & \cdots & k=K \\ (i,j)\in A & (i,j)\in A & & & (i,j)\in A \end{array} & \\ \begin{bmatrix} \mathcal{N} & 0 & \cdots & \cdots & 0 \\ -\mathcal{N} & 0 & \cdots & \cdots & 0 \\ 0 & \mathcal{N} & 0 & \cdots & 0 \\ 0 & -\mathcal{N} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \mathcal{N} \\ 0 & 0 & \cdots & \cdots & -\mathcal{N} \\ I & I & \cdots & \cdots & I \end{bmatrix} & \begin{array}{c} \forall i \in N \\ \forall i \in N \\ \forall i \in N \\ \forall i \in N \\ \vdots \\ \forall i \in N \\ \forall i \in N \\ \forall (i,j) \in A \end{array} \end{array}, \tag{1.4}$$

in which $\mathcal{N}$ is the n by m incidence matrix of the network G and I is the m by m identity matrix. Also b is a column vector with $2.n.K + m$ components as follows:

$$b = \begin{bmatrix} b^1 \\ -b^1 \\ b^2 \\ -b^2 \\ \vdots \\ b^K \\ -b^K \\ u \end{bmatrix} \begin{array}{c} k=1 \\ k=1 \\ k=2 \\ k=2 \\ \vdots \\ k=K \\ k=K \\ (i,j)\in A \end{array}, \tag{1.5}$$

where $b^k = [b_1^k, b_2^k, \ldots, b_n^k]^t$ is supply–demand vector with respect to commodity k, for $k = 1, \ldots, K$, and x, c and d are column vectors with $m.K$ components including the values of $x_{i,j}^k$, $c_{i,j}^k$ and $d_{i,j}^k$, respectively. Also, it is assumed that S_1 is nonempty and $d^t x + \beta > 0$ for all x in S_1. In addition, we assume that the optimal solution of fractional multi-commodity flow problem (1.1) and (1.2) is bounded.

Note that the objective function $f(x)$ is a pseudo-linear (both pseudo-convex and pseudo-concave) function and any local optimal solution of problem (1.3) is also a global optimal solution [8]. For this problem, Chadha and Chadha [1] proposed a linear dual problem and investigated weak duality, strong direct duality and weak complementary slackness theorems (see [9], also). In [10] a network simplex algorithm and also a similar dual problem is proposed for this problem with a single commodity namely a fractional minimal cost flow problem. In continuation, Sherali [11] showed that the resulted theorems