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An effective Chebyshev tau meshless domain decomposition method based on the integration–differentiation for solving fourth order equations

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ABSTRACT

In this paper, we present a method, which combines the Chebyshev tau meshless method based on the integration–differentiation (CTMMID) with Domain Decomposition Method (DDM), and apply it to solve the fourth order problem on irregular domains. This method, i.e. CTMMID-DDM, is an improvement of our previous job. In early work, it shows that the CTMMID can solve the fourth order problems well in square domain, and leads to the condition number which grows like $\mathcal{O}(N^4)$, but it becomes worse when we apply it on irregular domains directly. DDMs extend the applicability of spectral methods to handle complex geometries and large-scale problems. Numerical results show that CTMMID-DDM works well, it circumvents the ill-conditioning problem, further attains a improvement in solution accuracy, and also makes us feasible to solve the problems with boundary layers.

1. Introduction

Fourth order Partial Differential Equations (PDEs) in bounded domains may arise in solid mechanics to model the transversal displacements of elastic plates, or in fluid mechanics to model the stream-functions for the Navier–Stokes equations of incompressible fluids. From a general point of view, the biharmonic equation has been and is still extensively studied in the mathematical literature, either concerning its analysis or its numerical resolution. The numerical study can be accomplished by Finite Difference Methods (FDMs), Finite Element Methods (FEMs) and other methods [1]. Recently, spectral methods [2–5] have become increasingly popular in the computation of continuum mechanics problems. In contrast with error estimates for FEMs or FDMs, the convergence rate of spectral approximations is only limited by the regularity of the underlying function. However, spectral methods with high accuracy generally lead to the full and ill-conditioned coefficient matrix, especially for the differential equations on irregular domains [3].

The method of using integrated polynomial basis or some equivalent ideas can be traced back to 1950s. In the Chebyshev polynomial computations, Clenshaw [6] observed that the formula for the integral of a Chebyshev polynomial involves just two polynomials while the double integral involves only three. The most complete archive of early work is Section 5 of Fox and Parker in 1968 [7]. The authors pointed out that Clenshaw's strategy not only creates sparsity, but also may improve accuracy. The same result is also obtained by Gottlieb and Orszag [8] in the discretization of a constant coefficient second

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order differential equation. Zebib [9] introduced the strategy of solving a differential equation in terms of the Chebyshev expansion coefficients of the highest derivative that appears in the equation rather than in terms of the coefficients of the function itself, and has also shown that by this strategy, the accuracy of Chebyshev–Galerkin solutions to nonlinear BVPs can be improved. In more recent times, Leslie Greengard is a very enthusiastic proponent of converting differential equations to integral equations, he and his colleagues have accomplished more or less the same job in solving various problems [10–14]. These approaches have been widely used in the solution of second order equations and one-dimensional initial BVPs [15–17]. In our early study, we have ever tried the Chebyshev tau meshless method based on the integration of the highest derivatives (CTMMHD) to solve the fourth order problems [18]. But if we extend CTMMHD to two-dimensional case directly, the resulting coefficients matrix is ill-conditioned unfortunately. To overcome this shortcoming, a numerical scheme Chebyshev tau meshless method based on the integration–differentiation (CTMMID) has been reported in [19] to solve two-dimensional biharmonic type problems. The starting point is the Chebyshev expansion of the mixed partial derivative u_{xxyy}, then the lower derivatives or higher derivatives in both directions through an integration–differentiation process. Numerical results showed that the CTMMID obtained much higher accuracy than the Chebyshev collocation method based on the differentiation (CDF). The significant feature of CTMMID is to reduce the condition number from $\mathcal{O}(N^8)$ to $\mathcal{O}(N^4)$ for the problems on the rectangle domain. We use the Domain Embedding Methods (DEMs) [20–24] to solve the problems on irregular domains. The idea of these DEMs is to approximate the solution to the original problem by the solution of an auxiliary problem in a fictitious regular domain where the irregular geometry is embedded. Unfortunately, the condition number becomes worse when we apply CTMMID directly. It may make the problems on irregular domains, for example, the L-shaped domain, prone to the roundoff error. To improve the accuracy, it is necessary to decompose the domain into a set of subdomains [25].

In the last decade, the development in applying Domain Decomposition Methods (DDMs) to some discretized methods has drawn the attention of many researchers in science and engineering [26]. The DDM proposed by Schwarz [27] to solve PDEs in complex regions in 1960s, has many advantages such as its diversification of mathematic description of the physical problems. The complex geometry is decomposed into several simple ones where the application of the convenient methods is feasible. Upon the concept of spatial decomposition, the DDMs can be classified into two categories: overlapping (Schwarz methods) and non-overlapping (substructuring methods) [25]. In this paper, we consider only the non-overlapping approach. The boundary conditions on the interfaces are unknown, which can be determined as part of the solution procedure [28]. The basic part of difference DDMs lies in the way to achieve an estimate of the boundary conditions on the interfaces that ensures the continuity of the solution and of its normal derivative across the interfaces. The DDMs combined with several discretized methods, e.g. Differential Quadrature method (DQM) [29,30], Radial Basis Functions (RBFs) [26,28,31,32], Chebyshev collocation method based on integral formulation (PIF) [25,33], have been intensively studied for applications to second order elliptic problems, while not many existing numerical results to the fourth order equations [34–37]. In 1993, Henrichs [34] is the first person to establish the theoretical analysis of DDMs for the pseudo-spectral approximations of one-dimensional fourth order equations. In this paper, he proposed the continuity of the solution was up to the third order derivatives on the interface boundaries, and proved the convergence. N.Mai et.al [37] present a multidomain Integrated Radial Basis Function collocation method (MD-IRBFN) based on the multiquadric (MQ) interpolation to solve the elliptic problems, and compared the numerical results with the differential one (MD-DRBFN). The construction of the interface system is based on the continuity of the second and third order normal direction derivatives at interface points, a set of five equations $\partial^2 u/\partial x^2, \partial^2 u/\partial y^2, \partial^3 u/\partial x^3, \partial^3 u/\partial y^3$ and the governing equation, at intersection points. However, this paper [37] did not aim at diving deep in solving fourth order problems and there was no numerical tests on irregular domains.

Now, the study of the DDM is necessary for the following reasons:

- The CTMMID on single global irregular domain usually results in ill-conditioned system to be inverted. When handling the large-scale problems, the ill-conditioning makes the accuracy of CTMMID suffer from roundoff errors, and in some cases, produces numerically singular solutions.
- When the solution of the problem has many locally very different behaviors, it is clearly much easier to find an analytical description of localized regions than to find a single description over its entire range [28], for example, the singular perturbation problems with boundary layers.

In this paper, we conjunct CTMMID with DDM, and apply CTMMID-DDM to solve fourth order problems on irregular domains and singular perturbed problems with boundary layers. The problem domain is decomposed into several non-overlapping subdomains. Transmission of information among the subdomains is carried out through the interface boundaries. With the boundary reduction technique, we just need to solve the linear system resulted from the constraint conditions ($\partial^2 u/\partial n^2$ and $\partial^3 u/\partial n^3$, n-the normal direction) imposed on the interfaces, it highly reduces the computing efforts. Compared with MD-IRBFN, we do not have to give special attention to the continuity of the approximation solution at each intersection point, since CTMMID-DDM is 'pure' spectral method (expansion coefficients as unknowns). Numerical results show not only the improvement in the condition numbers does improve the accuracy, but also CTMMID-DDM is effective for the problems with boundary layers.

The remainder of the paper is organized as follows: in the next section, a brief view of CTMMID is given. Section 3 describes the domain decomposition technique for fourth order equations. Numerical examples including nonlinear problems, the biharmonic equation on L-shape domain, a singularly perturbed problem with boundary layers are presented in Section 4. A short discussion is given in the last section.