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An ℝ-linear conjugation problem for a plane two-component heterogeneous structure with an array of periodically distributed sinks/sources

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ABSTRACT

An ℝ-linear conjugation problem for a planar structure consisting of an isotropic strip and adjacent half-plane with contrasting permeabilities is solved. The whole structure is bounded from above by an equipotential line. An exact analytical solution is derived in terms of complex velocity in the class of one-periodical piece-wise meromorphic functions. Their principal part is the sum of periodically distributed simple poles, fixed in advance. Cases with singularities internal to the strip and subjacent half-plane are distinguished from a special case of poles positioned along the interface.

1. Introduction

Mathematical sinks and sources (poles) model horizontal, vertical wells and drains placed in rock/soil, with applications in groundwater hydrology, reservoir and agricultural engineering (see e.g. [1–4]). The generated fields of flow velocities depend on the number of wells, the distance between them, pumping/injection rates, hydraulic heads/pressures within borehole and the heterogeneity of the near-well zone (formation damage, skin effect [5]) and of the permeability of geological strata (soil layers) in which the wells are placed. The case of installing a drain or well in the substratum of a two-layered porous system has been studied in [6,1,7]. Here we extend their solutions to a periodic drainage. Moreover, in smartly designed bi-level agricultural drainage schemes [8] two arrays of drains are constructed such that within one period the upper drain commands over the lower drain. In petroleum industry the injection-abstraction wells are drilled also arbitrarily with respect to strata boundaries. So, in this paper we consider the most general case of an arbitrary number of sinks and sources positioned arbitrarily with respect to the interface between an upper stratum and substratum and an equipotential from which the fluid is supplied to the wells (ponded soil surface, quasi-horizontal water table or a quasi-static oil–water contact boundary in secondary recovery). The singularities (wells) can make periodic arrays (two as in [8] or any other number).

Thus, the present paper is a mathematical generalization of a recent work [9] where periodic drains in a two-layered soil were studied, with a limitation that the sinks are placed in the upper stratum just under an equipotential horizon (ponded soil surface). Similarly to the scheme in [9] a half–strip makes one period of the whole network of singularities. The sinks/sources are, however, arbitrary situated within this half-strip. In other words, a finite group of sinks/sources is infinitely but $2L$-periodically repeated as is shown in Fig. 1. Our aim is to find an exact analytical solution of the problem and using it to reconstruct a flow net in the structure. It is worth to say that at the present moment there are few works where conservative fields generated by a set of sinks, sources, vortices, dipoles and multipoles are tackled in an exact

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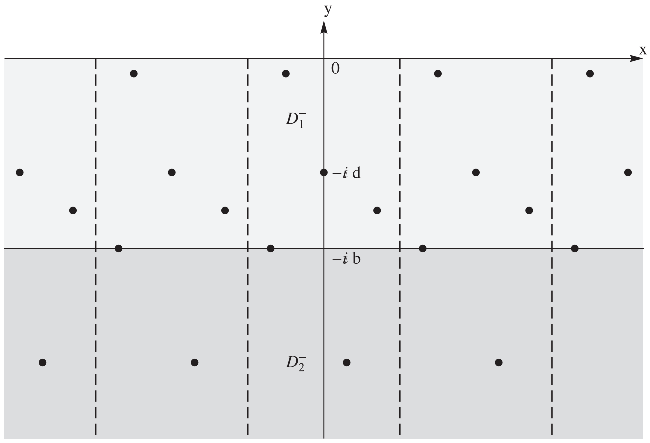

Fig. 1. 2-L periodically distributed set of sinks/sources.

analytical form [10–18]. In all these works the number of singularities of a complex potential was finite. Although in this paper the potential fields are interpreted for applications in porous media flows, mathematically similar models with singularities emerge, for instance in electrostatics and magnetostatic fields.

The article is organized as follows: the problem statement is provided in Section 2; the closed-form solutions are derived in Sections 3 and 4 for a unique singularity and a finite set of singularities arbitrary located inside the half-strip of periodicity, respectively; Section 5 discusses the results.

2. Formulation

In the vertical cross-section of Fig.1 we introduce a complex physical coordinate $z = x + \mathrm{i}y$. The complex potentials, $w_{1,2}(z) = \varphi_{1,2}(x,y) + \mathrm{i}\psi_{1,2}(x,y)$, in the two zones: $D_1 = \{z; -b < \mathrm{Im}z < 0\}$ and $D_2 = \{z; -\infty < \mathrm{Im}z < -b\}$ are defined as $\varphi_{1,2}(x,y) = -k_{1,2}h_{1,2}(x,y)$, where $h_{1,2}(x,y)$ is a hydraulic head and k_j is a hydraulic conductivity of the zone D_j. The functions $\psi_{1,2}(x,y)$ are harmonic conjugate with $\varphi_{1,2}(x,y)$. The boundary, coinciding with the real axe, is an equipotential line, i.e. $\varphi_1(x) \equiv 0$ there. Along the interface line $\mathcal{L} = \{z : z = x - \mathrm{i}b, -\infty < x < \infty\}$ usual boundary conditions hold: continuity of the stream functions and linear proportionality of the potential functions, i.e.

$$\psi_1(x - \mathrm{i}b) = \psi_2(x - \mathrm{i}b), \quad k_2\varphi_1(x - \mathrm{i}b) = k_1\varphi_2(x - \mathrm{i}b),$$

respectively.

The sinks and sources are placed arbitrary but $2L$-periodically in the upper layer D_1, lower half-space D_2, or at the interface $\mathcal{L}$. As is well-known (Polubarinova-Kochina, 1977) the functions $w_j'(z) = v_j(z) = v_{jx}(x,y) - \mathrm{i}\,v_{jy}(x,y)$ $(j = 1,2)$ (complex conjugated with complex velocity $\mathbf{v}_j(z) = v_{jx}(x,y) + \mathrm{i}\,v_{jy}(x,y)$) are holomorphic in D_j and continuous up to the boundary ∂D_j everywhere, except the singular points z_k, where the corresponding function $v_j(z)$ has a simple pole with a fixed residue. Besides, due to geometrical periodicity of distribution of the sinks/sources the required characteristic function must be periodical, i.e. $v_j(z + 2L) \equiv v_j(z)$. Additionally, we suppose that at infinity $v_2(z)$ vanishes.

We consider a half–strip of periodicity, $D^- = \{z : |\mathrm{Re}z| < L, \mathrm{Im}z < 0\}$, divided by the segment $[-L - \mathrm{i}b, L - \mathrm{i}b]$ into the rectangle $D_1^- = \{z : |\mathrm{Re}z| < L, -b < \mathrm{Im}z < 0\}$ and the half–strip $D_2^- = \{z : |\mathrm{Re}z| < L, \mathrm{Im}z < -b\}$. First, we assume that there is only one sink/source in the domain D^-.

3. The case of a single sink/source in the domain D^-

The above given boundary conditions could be written in terms of complex velocities in the following equivalent complex form [19, p.53]:

$$\begin{array}{ll} (1+\lambda)v_1(t) = v_2(t) - \lambda\overline{v_2(t)}, \quad t = x - \mathrm{i}b, & -L < x < L, \\ \mathrm{Re}\,v_1(x) = 0, & \end{array} \tag{1}$$

where

$$\lambda = \frac{k_2 - k_1}{k_2 + k_1}. \tag{2}$$

Let a required solution of the problem (1) has only one singular point. Without any loss of generality, we suppose that it is located on the symmetry axis of the domain D^-, i.e. $z_1 = -\mathrm{i}d$ $(d > 0)$. The principal part of $v(z)$ is