Applied Mathematical Modelling 37 (2013) 9338–9354

Contents lists available at SciVerse ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Simulation of the transverse vibrations of a cantilever beam with an eccentric tip mass in the axial direction using integral transforms

Carlos Frederico T. Matt *

Electric Power Research Center (CEPEL), Department of Lines and Substations, Cidade Universitária, Avenida Horácio Macedo 354, P.O. Box 68007, 21941-590 Rio de Janeiro, RJ, Brazil

A R T I C L E I N F O

Article history:
Received 5 March 2012
Received in revised form 26 October 2012
Accepted 29 April 2013
Available online 10 May 2013

Keywords:
Mechanical vibrations
Cantilever beam
Tip mass
Integral transform

A B S T R A C T

The main purpose of the current work is to employ an integral transform approach based on eigenfunction expansion and on an implicit filter scheme in order to solve the governing equations for the transverse vibrations of a cantilever beam clamped at one end and with an eccentric tip mass in the axial direction at the other end. Numerical results are obtained for both the undamped and damped natural frequencies of the system, as well as for its transverse displacement due to arbitrarily time-varying load and imposed displacement at the clamped end. The numerical results reported in the current work are highly accurate and new in the literature. New exact results are also provided for the transient displacement and its higher-order spatial derivatives to allow computation of bending stresses and strains. The relative merits of the proposed approach are finally pointed out.

1. Introduction

The mechanical vibrations of a cantilever beam carrying a concentrated mass at the tip have attracted much attention from scientists and engineers for a long time. The majority of previous works has focused on deriving and solving the exact frequency equation for the particular case of a concentrated mass and/or moment of inertia at the tip; see, for example, the earlier works of Prescott [1], Temple and Bickley [2], Pipes [3], Durvasula [4], Rama Bhat and Wagner [5] and To [6], to name just a few. The problem of mechanical vibrations on a cantilever beam with a tip mass continues to attract the attention of the research community due to a wide range of practical situations for which such mechanical system is a reasonably idealization. For instance, it may be used to investigate flexible robot arms [7], mast antenna structures [6,7], wind tunnel stings carrying an airplane or missile model [5], large aspect ratio wings carrying heavy tip tanks [5], Stockbridge dampers used for damping out aeolian vibrations on high-voltage transmission lines [8] or launch vehicles with payloads at the tip [5].

For some of the aforementioned applications, the concept of a concentrated mass or moment of inertia is violated because the tip mass is eccentric in the axial direction. By an eccentric tip mass in the axial direction one means a tip mass whose center of gravity lies along the beam neutral axis but it does not coincide with the attachment point of the tip mass to the beam. In order to account for an eccentric tip mass, the problem has been reformulated in a more general manner. Rama Bhat and Wagner [5] were the first to derive the exact frequency equation for a cantilever beam with a tip mass eccentric in the axial direction. Using a perturbation scheme, those authors provided approximate numerical results for the first five natural frequencies of the system as a function of two dimensionless parameters related to the mass and moment of inertia of the

* Tel.: +55 21 2598 6141; fax: +55 21 2270 4189.
E-mail address: cfmatt@cepel.br

0307-904X/$ - see front matter
http://dx.doi.org/10.1016/j.apm.2013.04.038

beam and the tip mass. The numerical results provided by Rama Bhat and Wagner [5] are highly accurate for small ratios between the distance of tip mass center of gravity to attachment point and the beam length. However, as the eccentricity of the tip mass increases (to be rigorously defined later on), the approximate solution derived by Rama Bhat and Wagner [5] rapidly looses accuracy. Six years later, To [6] addressed the same problem and derived the same frequency equation, although the boundary conditions at the tip provided in Ref. [6] be incorrect. The frequency equation was numerically solved by To [6] with the so called *regula falsi* or false position method. To [6] also provides an analytical expression for the frequency response function of the system when an harmonic displacement is imposed at the clamped end. Nevertheless, to the knowledge of the author, previous works neither take damping into account nor provide vibration results in both time and frequency domains.

Nowadays, owning to increasing trend of computing power, analytical schemes have been reconsidered as important tools, mainly when coupled with well-established numerical schemes. Integral transforms based on eigenfunction expansion is one of such analytical tools with many desirable features. It has been successfully applied in many structural vibration, heat and mass transfer problems with different names as modal analysis, modal superposition, classical and generalized integral transforms. There are excellent books devoted to the rigorous mathematical treatment behind integral transforms based on eigenfunction expansion for the solution of many practical problems; see, for instance, Refs. [9–14]. The extensive references cited therein also provide an excellent literature survey for readers interested in the mathematical theory behind such analytical–numerical scheme.

The main purpose of the current work is to employ an integral transform approach based on eigenfunction expansion and on an implicit filter scheme in order to solve the governing equations for the transverse vibrations of a cantilever beam clamped at one end and with an eccentric tip mass in the axial direction at the other end. In that way, one expands the field of application of the technique to an important problem in structural vibrations. To the knowledge of the author, no previous works have applied the implicit filter scheme to solve the aforementioned structural vibration problem. Numerical results are obtained for both the undamped and damped natural frequencies of the system, as well as for its transverse displacement due to arbitrarily time-varying load and imposed displacement at the clamped end. The numerical results reported in the current work are critically compared against previously reported benchmark solutions for the same problem. New results are also provided for higher-order spatial derivatives of the displacement to allow computation of bending stresses and strains and the relative merits of the proposed hybrid scheme are finally pointed out.

The present work can be viewed as an extension of previous works dealing with a similar mechanical system in two important directions. First, it provides accurate results for the *damped* natural frequencies for a wider range of tip mass eccentricity ratio. Second, it also provides exact results for the transverse displacement and its higher-order spatial derivatives under arbitrarily time-varying loads. These results may be used as benchmarks for purposes of comparison and verification of other analytical and numerical schemes.

2. Mathematical formulation of the physical problem

A sketch of a uniform cantilever beam carrying a rigidly mounted mass is shown in Fig. 1. The center of gravity of the tip mass may lie either to the right or to the left of the attachment point. In Fig. 1, the center of gravity is shown to the right of the attachment point. The governing partial differential equation for the small transverse vibrations of the cantilever beam is given by

$$EI\,\frac{\partial^4 w(x,t)}{\partial x^4}+\mu\,\frac{\partial^2 w(x,t)}{\partial t^2}+\eta\,\frac{\partial w(x,t)}{\partial t}=f(x,t), \tag{1}$$

where $f(x,t)$ stands for an arbitrarily time-varying external excitation; $w(x,t)$ denotes the transverse displacement of the beam at position x and time t; EI and μ denote, respectively, the bending or flexural stiffness and mass per unit length of the beam. The third term on the left-hand side of Eq. (1) stands for an aerodynamic damping of viscous type and the parameter η is the equivalent aerodynamic damping coefficient. The transverse displacement $w(x,t)$ is subjected to the following boundary-conditions at the clamped end $x=0$ and at the tip $x=L$, where L denotes the beam length (see Rama Bhat and Wagner [5]):

$$w(0,t)=f_0(t)\quad \text{at } x=0, \tag{2}$$

$$w'(0,t)=0\quad \text{at } x=0, \tag{3}$$

$$EI\,w''(L,t)=-(J_0+M\ell^2)\,\ddot{w}'(L,t)-M\ell\,\ddot{w}(L,t)\quad \text{at } x=L, \tag{4}$$

$$EI\,w'''(L,t)=M\,\ddot{w}(L,t)+M\ell\,\ddot{w}'(L,t)\quad \text{at } x=L. \tag{5}$$

In Eqs. (2)–(5), $f_0(t)$ denotes the imposed transverse displacement at the clamped end; M, J_0 and ℓ denote, respectively, the tip mass, the mass moment of inertia with respect to its center of gravity and the distance between the tip mass center of gravity and its point of attachment to the cantilever beam (henceforth, referred to as the eccentricity in the axial direction). Boundary conditions given by Eqs. (4) and (5) represent the equilibrium of bending moment and shear force at the tip. It is