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Identification of a spacewise dependent heat source

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ABSTRACT

The inverse problem of determining the temperature of a heat conductor together with an unknown spacewise dependent heat source from measured final data or time-average temperature observation is studied. The weak solution theory is applied for calculating the gradient of the least-squares functional that is minimized. For the general case when the heat source is the product between a known function $h(x,t)$ and the unknown source function $f(x)$ new explicit formulae, derived via the solution of the corresponding adjoint problem, are obtained. Numerical results obtained using the conjugate gradient method are presented and discussed.

1. Introduction

This paper is concerned with the mathematical analysis of the inverse spacewise dependent source problem with final “upper-base” or time-average temperature data. Mathematical models related to inverse problems of this type arise in various physical and engineering settings, e.g., the identification of sources of water and air pollution in the environment, or the determination of heat sources in heat conduction.

Although much of the analysis presented herein is applicable to general second-order linear PDEs with time-independent coefficients using the semigroup theory, see [1], for simplicity, we present the development of the theory for the classical one-dimensional heat equation.

We study the inverse problem of determining the temperature $u(x,t)$ and the heat source $f(x)$ in the parabolic heat equation

$$u_t(x,t) - u_{xx}(x,t) = f(x)h(x,t) + g(x,t), \quad (x,t) \in \Omega_T := (0,l) \times (0,T], \tag{1}$$

where h and g are given functions, and l and T are given positive constants. Here l represents the length of the finite heat conductor and the subscripts t and x in Eq. (1) denote the partial derivatives with respect to t and x, respectively. Eq. (1) has to be solved subject to the initial temperature condition

$$u(x,0) = u_0(x), \quad x \in [0,l], \tag{2}$$

the Dirichlet boundary conditions

$$u(0,t) = \mu_0(t), \quad u(l,t) = \mu_l(t), \quad t \in [0,T], \tag{3}$$

and the additional “upper-base” final temperature condition

$$u(x,T) = u_T(x), \quad x \in [0,l]. \tag{4}$$

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One can also consider Neumann or mixed boundary conditions instead of the Dirichlet boundary conditions (3).

Existence, uniqueness and stability results for the inverse source problem (1)–(4) are provided in [2–6], with numerical results presented in the case $h = 1$ in [7–9] and, more recently, in the case $h = h(t)$ in [10]. We note that by letting $h(x,t)$ multiply $f(x)$ it means that the direct method of [11], based on differentiating Eq. (1) with respect to t in order to eliminate the heat source $f(x)$, is not applicable. We finally mention that the existence of a solution of a related nonlinear inverse source problem in the case $h = h(u)$, instead of $h = h(x,t)$ in Eq. (1), has been investigated elsewhere, [12].

In comparison with previous works on the subject of spacewise heat source identification, we consider the case when $h = h(x,t)$ which is more general than [10] who considered only the case $h = h(t)$. Much of the detail of the proof of Theorem 2 is also different. Moreover, we shall also consider in Section 4, instead of (4) the following time-average temperature condition

$$\int_0^T u(x,t)dt = U_T(x), \quad x \in [0,l], \tag{5}$$

which has never been investigated numerically before and it represents one of the main contributions of the present study. Finally, we mention that we do not consider the general inverse problem investigated in [13,14] concerning the retrieval of the general source $f(x,t)$ from limited measurement data which do not ensure a unique solution.

The plan of the paper is as follows. In Section 2 we give the mathematical analysis of the inverse problem and recall previous existence and uniqueness results for solutions in Hölder and Hilbert spaces. In Sections 3 and 4 we introduce a quasi-solution of the inverse source problems (1)–(4) and (1)–(3), (5), based on the weak solution of the corresponding direct problem. Further, we introduce an adjoint parabolic problem and prove explicit relationships between the weak solution of this problem and the gradient of a cost objective functional which minimizes the L^2-gap norm between the computed and the given overspecified data. Based on these expressions, the conjugate gradient method is described in Section 5 for solving iteratively the inverse source problem with final or time-average temperature data observation. Section 6 presents and discusses the numerical results, whilst Section 7 gives the conclusions of the paper and possible future work.

2. Mathematical analysis

2.1. Strong solution in Hölder spaces

Let $\epsilon \in (0,1)$ be fixed and assume that:

(A) the functions $h, g \in H^{\epsilon,\epsilon/2}(\overline{\Omega}_T)$, $\mu_0, \mu_l \in H^{1+\epsilon/2}([0,T])$, $u_0 \in H^{2+\epsilon}([0,l])$ and the consistency conditions of order 0, namely, $\mu_0(0) = u_0(0), \mu_l(0) = u_0(l)$ are satisfied.

For the definition of the above Hölder spaces and norms, see [15]. Then, under assumption (A), there exists a unique solution $u \in H^{2+\epsilon,1+\epsilon/2}(\overline{\Omega}_T)$ of the direct problem (1)–(3) for any given source function $f \in H^{\epsilon}([0,l])$ such that the consistency conditions of order 1, namely,

$$\begin{cases} \mu_0'(0) - u_0''(0) = f(0)h(0,0) + g(0,0), \\ \mu_l'(0) - u_0''(l) = f(l)h(l,0) + g(l,0) \end{cases} \tag{6}$$

are satisfied, [15]. It is then natural to consider the Hölder space $(u,f) \in H^{2+\epsilon,1+\epsilon/2}(\overline{\Omega}_T) \times H^{\epsilon}([0,l])$ for the pair classical strong solution $(u(x,t), f(x))$ of the inverse source problem (1)–(4). However, if the source is sought in the general form $f(x,t)$ instead of $f(x)$, [13,14], then the solution of the inverse problem is not unique in general, [2], as one could easily add terms of the form $t(t-T)x(x-l)\chi(x,t)$, with arbitrary twice differentiable function χ, which satisfy the homogeneous form of conditions (2)–(4), to $u(x,t)$ and obtain a very different source function dependent on both space and time variables. Uniqueness of $f(x,t)$ can however be restored if one restricts the space of functions in which the source $f(x,t)$ is to lie, [16,14,17].

Let us assume now that the additional final observation (4) satisfies:

(B) $u_T \in H^{2+\epsilon}([0,l])$ and the following compatibility conditions:

$$\begin{cases} \mu_0'(T) - u_T''(0) = f(0)h(0,T) + g(0,T), \\ \mu_l'(T) - u_T''(l) = f(l)h(l,T) + g(l,T). \end{cases} \tag{7}$$

The assumption (B) is a consequence of (6) and the compatibility condition for the input data at $t = T$. Then, if the assumptions (A) and (B) are satisfied and if $|h(x,t)| \geqslant h_0 > 0$ for all $(x,t) \in \overline{\Omega}_T$, the inverse source problem (1)–(4) has at most one solution $(u,f) \in H^{2+\epsilon,1+\epsilon/2}(\overline{\Omega}_T) \times H^{\epsilon}([0,l])$, [3]. We also have the following stronger solvability result of [4].

Theorem 1. *Suppose that* $h, h_t, g \in H^{\epsilon,\epsilon/2}(\overline{\Omega}_T), u_0,\ u_T \in H^{2+\epsilon}([0,l]), \mu_0, \mu_l \in H^{1+\epsilon/2}([0,T])$,

$$h(x,t) \geqslant 0, \quad h_t(x,t) \geqslant 0, \quad |h(x,T)| \geqslant h_T > 0 \quad \forall (x,t) \in \overline{\Omega}_T,$$