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Bernstein operational matrix of fractional derivatives and its applications

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A B S T R A C T

In this paper, Bernstein operational matrix of fractional derivative of order α in the Caputo sense is derived. We also apply this matrix to the collocation method for solving multi-order fractional differential equations. The numerical results obtained by the present method compares favorably with those obtained by various collocation methods earlier in the literature.

1. Introduction

In recent years, there has been a great deal of interest in fractional calculus since there have been a wide variety of applications in physics and engineering (see for example [1] and the references therein). A number of definitions for the fractional derivative have emerged over the years while the Riemann–Liouville and Caputo definitions are the most commonly used ones. The Caputo definition of order $\alpha > 0$ is defined as

$$D^{\alpha}f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}dt, & n-1 < \alpha < n, \quad n \in \mathbb{N}, \\ \frac{d^n}{dx^n}f(x), & \alpha = n \in \mathbb{N}. \end{cases} \tag{1}$$

For the Caputo's derivative we have [2]

$$D^{\alpha}C = 0 \quad (C \text{ is a constant}), \tag{2}$$

$$D^{\alpha}x^{j} = \begin{cases} 0, & \text{for } j \in \mathbb{N}\cup\{0\} \text{ and } j < \lceil\alpha\rceil, \\ \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}x^{j-\alpha}, & \text{for } j \in \mathbb{N}\cup\{0\} \text{ and } j \geqslant \lceil\alpha\rceil \text{ or } j \notin \mathbb{N} \text{ and } j > \lfloor\alpha\rfloor. \end{cases} \tag{3}$$

We use the ceiling function $\lceil\alpha\rceil$ to denote the smallest integer greater than or equal to α, and the floor function $\lfloor\alpha\rfloor$ to denote the largest integer less than or equal to α. Caputo's fractional differentiation is a linear operation:

$$D^{\alpha}(c_1f_1(x) + c_2f_2(x)) = c_1D^{\alpha}f_1(x) + c_2D^{\alpha}f_2(x), \tag{4}$$

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where c_1 and c_2 are constants. There are different techniques for solving fractional differential equations, fractional integro-differential equations and fractional optimal control problems, like fractional characteristic method [3], variational iteration method [4,5], Adomian decomposition method [6], operational matrix of B-spline functions [7], operational matrix of Legendre polynomials [8,9], operational matrix of Chebyshev polynomials [10], Legendre collocation method [11], pseudo-spectral method [12], Legendre multiwavelet collocation method [13] and other methods [14–16].

In the present paper we intend to extend the application of the Bernstein polynomials to solve the fractional order differential equations. Our main aim is to generalize the Bernstein polynomials operational matrix to fractional calculus. It is worthy to mention here that, the method based on using the operational matrix for solving differential equations is computer oriented. The main characteristic behind the approach using this technique is that it reduces these problems to those of solving a system of algebraic equations thus greatly simplifying the problem. The paper is organized as follows. We begin by introducing some properties of Bernstein polynomials which are required for establishing our results. In Section 3 the Bernstein operational matrix of fractional derivative is obtained. Section 4 is devoted to applying the Bernstein operational matrix of fractional derivative for solving multi-order fractional differential equations. Numerical simulations are reported in Section 5. In Section 6, we give a brief conclusion.

2. Properties of Bernstein polynomials

The well known Bernstein polynomials of the nth degree are defined on the interval $[0,1]$ as [17,18]

$$b_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad i = 0,\ldots,n. \tag{5}$$

These Bernstein polynomials form a complete basis on over the interval $[0,1]$. A recursive definition also can be used to generate these polynomials

$$b_i^n(x) = (1-x)b_i^{n-1}(x) + x b_{i-1}^{n-1}(x), \quad i = 0,\ldots,n,$$

where $b_{-1}^{n-1}(x) = 0$ and $b_n^{n-1}(x) = 0$. Since the power basis $\{1, x, x^2, \ldots, x^n\}$ forms a basis for the space of polynomials of degree less than or equal to n, any Bernstein polynomial of degree n can be written in terms of the power basis. This can be directly calculated using the binomial expansion of $(1-x)^{n-i}$, one can show that

$$b_i^n(x) = \sum_{j=i}^{n} (-1)^{j-i} \binom{n}{i} \binom{n-i}{j-i} x^j, \quad i = 0,\ldots,n. \tag{6}$$

On the other hand, the fact that they are not orthogonal turns out to be their disadvantage when used in the least-squares approximation. As said in [19] one approach to direct least-squares approximation by polynomials in Bernstein form relies on construction of the basis $\{d_0^n(x), d_1^n(x), \ldots, d_n^n(x)\}$ that is "dual" to the Bernstein basis of degree n on $x \in [0,1]$. This dual basis is characterized by the property

$$\int_0^1 b_i^n(x) d_j^n(x) dx = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j, \end{cases}$$

for $i, j = 0, 1, \ldots, n$. A function $f(x)$, square integrable in $[0,1]$, may be expressed in terms of the Bernstein basis [17,18]. In practice, only the first$(n+1)$ term Bernstein polynomials are considered. Hence if we write

$$f(x) \simeq \sum_{i=0}^{n} c_i b_i^n(x) = C^T B(x), \tag{7}$$

where the Bernstein coefficient vector C and the Bernstein vector $B(x)$ are given by

$$C^T = [c_0, \ldots, c_n],$$

$$B(x) = [b_0^n(x), b_1^n(x), \ldots, b_n^n(x)]^T, \tag{8}$$

then

$$c_i = \int_0^1 f(x) d_i^n(x) dx, \quad i = 0, 1, \ldots, n.$$

Author of [20] has derived explicit representations

$$d_j^n(x) = \sum_{k=0}^{n} \lambda_{jk} b_k^n(x), \quad j = 0, 1, \ldots, n,$$

for the dual basis functions, defined by the coefficients