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Elastic solutions of a functionally graded cantilever beam with different modulus in tension and compression under bending loads

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ABSTRACT

This paper considers a functionally graded cantilever beam with different modulus in tension and compression. The beam is subjected to bending loads, including pure bending, shear force at the free end and uniform pressure on the upper lateral, respectively. Its modulus values in tension and compression both change with the thickness coordinate as arbitrary functions, which could bring the beam a broader range of applications in engineering. The problem is treated as a plane stress case and described by Airy stress function. By using semi-inverse method, the elastic solutions for the beam are obtained, which can be easily degenerated into the ones for homogeneous beams. An example is finally presented to show the effect of nonhomogeneous materials with different modulus on the elastic field in a cantilever beam.

1. Introduction

It is known that many materials exhibit different elastic modulus in tension and compression. These materials mentioned first by Timoshenko [1] are widely used in engineering practice. During the last decade, a number of theories have been produced for these materials, while two main criterions are universally accepted: (1) the criterion of positive–negative signs in the longitudinal strain of fibers proposed by Bert [2] for laminated composites [3–5]; (2) the criterion of positive–negative signs of principal stress developed by Ambartsumyan [6] for isotropic materials. Yao and Ye [7] derived an analytical solution to bending beam, based on determination of neutral surface of structures. Ye and Chen [8] proposed a new finite element formulation for planar elastic deformation. Babeshko and Shevchenko [9] presented a numerical technique for the elastoplastic and thermoelastoplastic stress–strain state of flexible layered shells under axisymmetric loading. He et al. [10] obtained an analytical solution to bending thin plates with different modulus.

Recently, to meet the requirements of engineering practice, plenty of new materials with strong, stiff and light properties are invented in structural design, especially functionally graded materials (FGMs). FGMs have no distinct internal boundaries and their properties change smoothly with respect to the spatial coordinate, which makes them promising in engineering applications. Thus, many studied have been performed on the mechanical behavior of FGMs and their laminated structures. Sankar [11] obtained an elastic solution of a functionally graded beam subjected to a transverse sinusoidal load, in which elastic modulus exhibits exponential variation through the thickness. Zhong and Yu [12] presented a general solution of a functionally graded beam with arbitrary graded variations of material property by the Airy stress function. Zhu and Sankar

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[13] solved the 2D elastic equations for a FGM beam subjected to transverse loads by means of combined Fourier series-Galerkin method, in which the variation of Young's modulus through the thickness was given by a polynomial in the thickness coordinate and Poisson's ratio was assumed to be constant.

Now, many studies have been made on structures using homogeneous materials with different modulus and functionally graded materials, respectively, such as concrete beams, composite plates and metal-ceramic packages. However, few researches has involved FGM structures with different modulus in tension and compression. It might restrict applications of this new form structures in engineering. In this paper, a FGM cantilever beam with different modulus carrying a pure bending moment, a concentrated shear force at the free end and a uniform pressure on the upper lateral, respectively, are considered, and its elastic solutions are investigated.

2. Problem description and analytic model

A general model of a FGM cantilever beam with different modulus is shown in Fig. 1. A Cartesian coordinate system is introduced into the analytical model. The length of the beam is l and the thickness is h.

The elastic modulus of tension and compression vary with the thickness coordinate as arbitrary functions $F^+(y)$ and $F^-(y)$, respectively, and Poisson's ratio holds constant.

In the absence of body forces, the basic equations for plane stress static problem include the equations of equilibrium, strain–displacement relations as follows

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}=0,\quad \frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0, \tag{1}$$

$$\varepsilon_x=\frac{\partial u}{\partial x},\varepsilon_y=\frac{\partial w}{\partial y},\quad \gamma_{yx}=\frac{\partial u}{\partial y}+\frac{\partial w}{\partial x}. \tag{2}$$

Symbols ε_x, ε_y and γ_{yx} are the normal and shear strain components. Symbols $u$andwdenote the displacement components. The strain compatibility equation can be derived from Eq. (2)

$$\frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2}-\frac{\partial^2 \gamma_{yx}}{\partial y \partial x}=0. \tag{3}$$

The constitutive relations of FGMs are

$$\varepsilon_x=s_{11}\sigma_x+s_{12}\sigma_y,\quad \varepsilon_y=s_{12}\sigma_x+s_{22}\sigma_y,\quad \gamma_{yx}=s_{44}\tau_{yx}, \tag{4}$$

with

$$s_{ij}=\begin{cases} s_{ij}^{+}=s_{ij}^{0+}F^{+}(y) \\ s_{ij}^{-}=s_{ij}^{0-}F^{-}(y)\end{cases}\quad (i=1,2), \tag{5}$$

where s_{ij}^+ and s_{ij}^- are tension and compression elastic compliance parameters. Comparing with homogeneous materials, the elastic compliance parameters are arbitrary functions of thickness coordinate instead of constant values. Symbols s_{ij}^{0+} and s_{ij}^{0-} are constants, and represent the basic elastic parameters of FGMs in tension and compression at reference plane $y=0$. Here, $F^+(y)$ and $F^-(y)$ are graded functions of FGMs in tension and compression, respectively.

In order to satisfy the equations of equilibrium, the stress components are defined in terms of Airy stress function $\varphi_i(x,y)$

$$\sigma_x=\frac{\partial^2 \varphi}{\partial y^2},\quad \sigma_y=\frac{\partial^2 \varphi}{\partial x^2},\quad \tau_{yx}=-\frac{\partial^2 \varphi}{\partial y \partial x}. \tag{6}$$

By substituting Eq. (4) and Eq. (6) into Eq. (3), we can obtain the governing equation for Airy stress function $\varphi(x,y)$

$$\frac{\partial^2}{\partial y^2}\left(s_{11}\frac{\partial^2 \varphi}{\partial y^2}+s_{12}\frac{\partial^2 \varphi}{\partial x^2}\right)+\frac{\partial}{\partial y}\left(s_{44}\frac{\partial^3 \varphi}{\partial x^2 \partial y}\right)+s_{12}\frac{\partial^4 \varphi}{\partial y^2 \partial x^2}+s_{22}\frac{\partial^4 \varphi}{\partial x^4}=0. \tag{7}$$

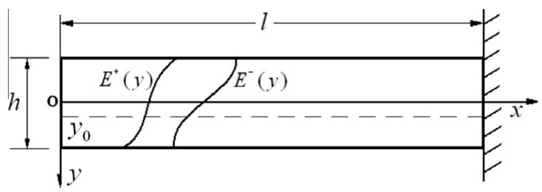

Fig. 1. The model for the cantilever FGM beam with different modulus in tension and compression.