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Moment independent sensitivity analysis with correlations

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ABSTRACT

The moment independent importance measure is a popular global sensitivity analysis technique, and aims at evaluating contributions of the inputs to the whole output distribution. In this work, moment independent sensitivity analysis is performed for models with correlated inputs, by decomposing the importance measure into the uncorrelated part and correlated part. The correlated input variables are first orthogonalized, then the moment independent sensitivity analysis of the newly generated independent variables is performed. Discussions indicate that the moment independent importance measures, so obtained, can be interpreted as the full, correlated and uncorrelated contributions of the original inputs to the whole output distribution. Procedure of the proposed approach has been generalized. By the decomposition, the information provided by the moment independent importance analysis is enriched, which has been demonstrated by the application to several examples.

1. Introduction

Sensitivity analysis (SA) is to study "*how uncertainty in the output of a model (numerical or otherwise) can be apportioned to different sources of uncertainty in the model input factors*" [1]. Generally, SA can be classified into two main branches [2]: local sensitivity analysis (LSA), which is often carried out in the form of derivative of the model output with respect to the input parameters; and global sensitivity analysis (GSA), focusing on the output uncertainty over the entire range of the inputs. The limitation of LSA, as a derivative based approach, lies in the fact that derivatives are only informative at the base points where they are calculated and do not provide for an exploration of the rest of the input space. GSA, on the other hand, explores the whole space of the input factors, thus is more informative and robust than estimating derivatives at a single point of the input space. Obviously, GSA has a greater potential in engineering applications.

Global sensitivity indices are also known as importance measures, and have attracted abundant research interests in the literature. The family of importance measures generally includes nonparametric techniques suggested by Saltelli and Marivoet [3], and Iman et al. [4], variance based methods suggested by Sobol [5], and further developed by Rabitz and Alis [6], Saltelli et al. [7], Frey and Patil [8], and moment-independent approaches proposed by Park and Ahn [9], Chun et al. [10], Borgonovo [11,12], Liu and Homma [13], Castaings et al. [14]. It has been found that the nonparametric techniques are insufficient to capture the influence of inputs on the output variability for nonlinear models and also when interactions among inputs emerge [8]. As Saltelli [1] underlined, an importance measure should satisfy the requirement of being "*global, quantitative, and model free*", and he advocated that the variance based importance measure is a preferred way of measuring uncertainty importance. However, the use of variance as an uncertainty measure relies on the assumption that "*this moment*

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is sufficient to describe output variability" [1]. Borgonovo [11] showed that relying on the sole variance as an indicator of uncertainty would sometimes lead the decision maker to noninformative conclusions, since the inputs that influence variance the most are not necessarily the ones that influence the output uncertainty distribution the most. Borgonovo [11] extended Saltelli's three requirements by adding "*moment independent*", and proposed a new importance measure which looks at the influence of input uncertainty on the entire output distribution without reference to a specific moment of the output.

Most of the existing techniques for importance analysis are performed under the hypothesis of input independence. However, in many real cases the input factors are correlated with one another [15–17]. Although researchers may take measures to "reasonably" neglect the correlation for sake of computability, still it may have significant impacts on the importance results. And for the studies in the literature concerning the importance analysis with correlated input factors, there also exists a question: researchers just find the contribution of one input factor to the output uncertainty, with no knowledge of what the contribution is composed of [17]. As pointed out by Xu and Gertner [17], it is necessary to divide the contribution of uncertainty to the output by an individual input factor into two parts: the uncorrelated part, which means this part is completely immune from the other input factors and is produced by this input factor "individually and independently", and the correlated part, which means this part is produced by the correlation of this input variable with the others. This knowledge can give us a better understanding of the composition of the output uncertainty, and decide whether the uncorrelated part or the correlated part should be focused on. It can be also noticed that the extensive research related to sensitivity analysis with correlations by now is mainly focusing on the variance based importance measures. Over the past ten years or so many methods have been developed to generalize variance based SA for the case of correlated input variables [15–17], however, this issue still remains pendent and ambiguous, which is mainly due to the fact that the variance decomposition under correlation is not unique and clear as the Hoeffding-Sobol decomposition with independent inputs [11,18]. This can be seen as an intrinsic flaw of the variance based importance measures.

In this work, we treat the importance analysis of models with correlated inputs from a brand new angle. As Borgonovo pointed out [11], the moment independent importance measure can fully consider the analysts state of belief of the problem by looking at the entire output distribution. Besides, its definition has no relationship with the Hoeffding-Sobol decomposition, thus it will not be hassled by the problem met by the variance based importance measures when dealing with correlated input variables. With this advantage, we consider the contribution of the inputs to the output distribution with correlations by using the moment independent measure. For models with correlated inputs, Borgonovo's importance measure will be decomposed into two parts: the uncorrelated contribution and the correlated contribution. By this decomposition, the information provided by the moment independent importance analysis will be enriched. To this aim, correlated input variables will be first decorrelated with the Gram–Schmidt procedure, which is also adopted by Mara and Tarantola [19]. Then the moment independent importance measures of the newly generated independent variables are calculated, which is rather straightforward. It has been shown that the moment independent importance measures, so obtained, can be interpreted as the full, correlated and uncorrelated contributions of the original inputs to the whole output distribution.

The remainder of the paper is organized as follows. Section 2 reviews the definition of the moment independent importance measure proposed by Borgonovo. In Section 3, we will detail the approach to decompose the moment independent importance measure for models with correlated input variables. In Section 4, several examples are presented to demonstrate the feasibility of the proposed approach. Finally, conclusions of this work are highlighted in Section 5.

2. Review on Borgonovo's importance measure

Suppose that the input–output model is denoted as $Y = g(\boldsymbol{X})$, where Y is the model output, and $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ (n is the model dimension) is the vector of random input variables. The uncertainties of the input variables are represented by given probability distributions. The joint probability density function (PDF) of $\boldsymbol{X}$ is denoted as $f_{\boldsymbol{X}}(\boldsymbol{x})$, and the marginal PDF of X_i can be formulated as

$$f_{X_i}(x_i) = \int \cdots \int f_X(\boldsymbol{x}) \prod_{k=1,k\neq i}^{n} \mathrm{d}x_k. \tag{1}$$

The unconditional PDF of Y is denoted as $f_Y(y)$. Fix the input of interest X_i at its realization x_i, then we can obtain the conditional PDF of Y, denoted as $f_{Y|X_i}(y|x_i)$. $f_{Y|X_i}(y|x_i)$ can be obtained by

$$f_{Y|X_i}(y|x_i) = \frac{f_{X_i,Y}(x_i, y)}{f_{X_i}(x_i)}, \tag{2}$$

where $f_{X_i,Y}(x_i, y)$ is the joint PDF of X_i and Y.

Borgonovo [11] proposed his importance measure considering the contribution of inputs to the PDF of the model output. The shift between $f_Y(y)$ and $f_{Y|X_i}(y|x_i)$ is measured by the area $s(x_i)$ closed by the two PDFs (illustrated in Fig. 1), which is given by

$$s(x_i) = \int \left| f_Y(y) - f_{Y|X_i}(y|x_i) \right| \mathrm{d}y. \tag{3}$$