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Finite difference/spectral approximations to a water wave model with a nonlocal viscous term

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ABSTRACT

The numerical simulation of a water wave model with a nonlocal viscous dispersive term is considered in this article. We construct two linearized finite difference/spectral schemes for numerically solving the considered water model. A particular attention is paid to the treatment of the nonlocal dispersive term and the nonlinear convection term. The proposed methods employ a known $(2-\alpha)$-order scheme for the α-order fractional derivative and a mixed linearization for the nonlinear term. A detailed analysis shows that the proposed schemes are unconditionally stable. Some error estimates are provided to predict that the method using the linearized Euler plus $(2-\alpha)$-order scheme in time and the spectral approximation in space is convergent with order of $\mathcal{O}(\Delta t + N^{1-m})$, where Δt, N and m are, respectively the time step size, polynomial degree, and regularity in the space variable of the exact solution. Moreover, we prove that the second order backward differentiation plus $(2-\alpha)$-order scheme converges with order 3/2 in time. A series of numerical examples is presented to confirm the theoretical prediction. Finally the proposed methods are used to investigate the asymptotic decay rate of the solutions of the water wave equation, as well as the impact of different terms on this decay rate.

1. Introduction

The study of wave propagation in shallow water has a long history in many research fields. It is classical to model the small amplitude long waves by the Boussinesq systems and KdV equation [1] or its regularized version, the so-called BBM (Benjamin–Bona–Mahony) equation [2]. The BBM equation is regarded as an improvement of the KdV equation for modeling long surface gravity waves of small amplitude in 1 + 1 dimensions. It has been shown that the BBM equation possesses stability and uniqueness of solutions. This contrasts with the KdV equation, which is unstable in its high wavenumber components.

Kakutani & Matsuuchi [3] first pointed out that the asymptotic model for viscous water waves is an equation involving both standard diffusion and nonlocal pseudo-differential operators reflecting dispersive and diffusive effect stemming from the viscous layer in the fluid. Liu & Orfila [4] derived a set of depth-integrated equations for transient long-wave propagation with viscous effects included. In their model the viscous effect is represented by convolution integrals due to the diffusion process in the boundary layer. Chen & Goubet [5] have studied various dissipative mechanics associated with the Boussinesq

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systems which model two-dimensional small amplitude long wavelength water waves. Dutykh & Dias [6] introduced a system which models water waves in a fluid layer of finite depth under the influence of viscous effects. They obtained a visco-potential free-surface flows model involving local and nonlocal dissipative terms. The local dissipation terms come from molecular viscosity while the nonlocal dissipative term represents a correction due to the presence of a bottom boundary layer. In [7] Dutykh analyzed dispersion relation of the proposed models, and the effect of the nonlocal term on solitary and linear progressive waves attenuation was investigated with the long wave approximation.

The investigation of the decay rates and the effect of viscosity of solutions to these equations has been subject of many research efforts in recent years. Amick et al. [8] investigated long time behavior of the decay rate of solutions to Korteweg-de Vries equation and the regularized long-wave equation for any initial data. They found that the solution decays in time as $\mathcal{O}(t^{-1/4})$, $\mathcal{O}(t^{-3/4})$ and $\mathcal{O}(t^{-1/2})$ in the L^2-norm, H^1- semi norm, and L^∞- norm, respectively. Bona et al. [9] presented an asymptotic form which renders explicit the relative strengths of the dissipative and dispersive effects in the solutions that decay to zero as time tends to infinity. Bona et al. [10] introduced a Fourier splitting method for analyzing the long-time behavior of solutions of the generalized Korteweg-de Vries equation posed on the whole line.

Some other work have been focused on using numerical methods to simulate the behavior of the solutions of water wave equations. Chen [11] proposed a numerical scheme for initial and boundary value problems of a two-dimensional Boussinesq system which describes water waves over a moving bottom. The proposed method combines a semi-implicit finite difference scheme in time and spectral methods in space, which is conditionally stable. Chen et al. [12] were concerned with the well-posedness and numerical computation of the solution to a water wave equation and its simplifications (i.e., without second order term or third order term) with a nonlocal term. A method using a semi-implicit scheme for the time discretisation and Fourier approach in space was proposed to compute the decay rates. In their method the nonlinear term was treated explicitly while the others were treated implicitly. Goubet & Warnault [13] discussed a linear viscous asymptotic model for water waves, and provided some estimates for the decay rate of solutions towards the equilibrium. In a recent work [14], Dumont & Duval investigated the role of the non local viscous terms, the geometric dispersion and the nonlinearity in two asymptote models by using the Gear scheme. The Gear scheme for the discretization of the nonlocal term was also employed in [15,16] to investigate the long time effects or asymptotic regularization effects of the damping models.

In the paper, we will propose and analyze two finite difference/spectral schemes for the water wave equation in a bounded domain with the periodic boundary condition. The proposed schemes combine a linearized finite difference method in time and Fourier spectral method in space. The finite difference schemes used here for the time discretisation take the advantage that only a linear system is solved at each time step, and the computation is unconditionally stable. We give a detailed analysis of the proposed schemes by providing some stability and error estimates. Based on this analysis, we prove that the overall schemes are unconditionally stable, and the linearized Euler scheme in time/spectral method in space converges with the order of $\mathcal{O}(\Delta t + N^{1-m})$, while the second order backward differentiation has 3/2-order convergence in time, where Δt and N are, respectively the time step size and the space resolution, m is the regularity in the space variable of the exact solution. At last, the proposed methods are used to investigate the asymptotic decay rate of the solutions for the water model. The effect of the nonlocal term, the dispersion term, and the nonlinear term on the decay rate will be discussed. Since long time integration is needed in the investigation of the asymptotic decay rate, the numerical simulation greatly benefits from the unconditional stability of the schemes which allow use of large time step sizes.

The outline of this paper is as follows. In the next section we first briefly recall some basic known properties of the water wave equation, then construct two schemes for the time discretisation of this equation. A discrete energy inequality is established for each of these two schemes, showing that the proposed time schemes are unconditionally stable. In Section 3, we present the Fourier spectral method for the spatial discretisation and derive the error estimates for the full discrete problems. Some numerical results are presented in Section 4 which support the theoretical statement. We also analyze long-time behavior of the decay rate of the solutions with different values of various parameters. Finally, some concluding remarks are given in the final section.

2. Water wave models and discretisation in time

We consider the water wave equation in the BBM form as follows:

$$\partial_t u + \partial_x u - \beta\partial_{xxt} u + \frac{\nu^{1/2}}{\Gamma(1/2)}\int_0^t \frac{\partial_s u(s)}{(t-s)^{1/2}}ds + \gamma u\partial_x u - \alpha\partial_{xx} u = 0, \tag{2.1}$$

where β, ν, γ, and α are positive parameters. It is noted that in the usual BBM wave equation, α is set to be ν. Use of different parameters α and ν in this paper will enable us to investigate the effect of the nonlocal viscous term and the diffusion term separately. We recall below the existing results (see, e.g., [12]) on the global existence and the decay rate of the solution for the problem (2.1) with $\beta = 0$ and small initial datum.

Theorem 2.1. [12] *Consider the Eq. (2.1) with β = 0 supplemented with initial data $u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then there exists $\varepsilon > 0$, $c(u_0) > 0$ such that for all $\|u_0\|_{L^1(\mathbb{R})} < \varepsilon$, the Eq. (2.1) admits a unique global solution $u \in C(\mathbb{R}_+; L^2_x(\mathbb{R})) \cap C^1(\mathbb{R}_+; H_x^{-2}(\mathbb{R}))$. In addition, u satisfies*