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Analysis of $GI/M/n/n$ queueing system with ordered entry and no waiting line

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ABSTRACT

This paper deals with a multi-server, finite-capacity queuing system with recurrent input and no waiting line. The interarrival times are arbitrarily distributed whereas service times are exponentially distributed. Moreover, the servers are heterogeneous and independent of each other. Arriving customers choose the server with the lowest index number among the empty servers. When all servers are busy at a time of an arrival, that arrival must leave the system without being served. The semi-Markov process method is used to describe this model and embedded Markov chain of the process is obtained. Furthermore, the Laplace–Stieltjes transform of the distribution of interoverflow times is derived which is the main objective of the paper. Finally, it is offered a new formulation for the loss probability which provides more efficient and rapid calculation is proposed.

1. Introduction

The queueing models with no waiting line have been applied in many areas like telecommunication networks, design of call centers and emergency service systems. In the classical queueing system $M/M/n/n$ investigated first by Erlang [1], the probability of being in the state k is determined as follows:

$$P_k = \frac{\rho^k/k!}{\sum_{k=0}^{n}(\rho^k/k!)} \quad 0 \leqslant k \leqslant n, \tag{1}$$

where $\rho = \lambda/\mu$ is the offered load, λ^{-1} and μ^{-1} are the means of the interarrival times and service times, respectively. Formula (1) is known as Erlang loss formula for $k = n$, which is of great importance for the mathematical modeling of communication systems and has been a source of inspiration to analyze more complicated systems. The authors prefer to use $GI/G/n/0$ notation for the queuing model $GI/G/n/n$ with no waiting line in the rest of paper.

Konig and Matthes [2] investigated Erlang's aforementioned formula and generalized it for the system with dependent service times. Takacs [3] re-analyzed Erlang's model while considering arrival and departure times of the customers and used the discrete-parameter stochastic process method. Brumelle [4] generalized Erlang's model for dependent arrivals and service rates and obtained the mean waiting time of a customer in the system.

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Palm [5] extended the Erlang's model for the case of having independent interarrival times with an arbitrary distribution for the $GI/M/n/0$ queuing model. Besides, the stream of overflows was analyzed and the loss probability was computed as follows:

$$\frac{1}{P_n} = \sum_{k=0}^{n} \binom{n}{k} c_k, \tag{2}$$

where, with f being the Laplace–Stieltjes transform of the interarrival time, and c_k are

$$c_k = \begin{cases} 1, & k = 0 \\ \prod_{k=1}^{n} \frac{1-f(k\mu)}{f(k\mu)}, & 1 \leqslant k \leqslant n \end{cases} \tag{3}$$

Takacs [6] proved that the limit distribution of being in any state is independent of the initial state. Similar results were obtained when the number of servers was infinite, as well. Takacs [7] obtained Palm's loss formula (2) in a simpler way by using the method of finite difference equations. Moreover, Takacs [8] considered the number of customer immediately before the arrival of nth customer as a sequence of random variables $\{\eta_n\}(n = 1, 2, \ldots)$. Mentioned sequence of random variables is a Markov chain whose one-step transition probabilities $p_{ij} = P[\eta_{n+1} = j|\eta_n = i]$ are as follows:

$$p_{ij} = \binom{i+1}{j} \int_0^{\infty} e^{-j\mu t}\left(1 - e^{-\mu t}\right)^{i+1-k} dF(t), \tag{4}$$

for $j = 1, 2, \ldots, n-1$, and $p_{n,j} = p_{n-1,j}$, and $F(t)$ is the distribution of interarrival times. Takacs [8,9] obtained Palm's recurrence formula as follows:

$$f_k(s) = \frac{f_k(s+\mu)}{1 - f_{k-1}(s) + f_{k-1}(s+\mu)} \quad k = 1, 2 \ldots, \tag{5}$$

where $f_0(s) = f(s)$ is Laplace–Stieltjes transform of interarrival time distribution $F(t)$.

The interarrival and service times have been considered as arbitrarily distributed in the literature. Halfin [10] studied the $GI/G/1$ queueing model with no waiting line and obtained the distribution function of the interoverflow times of customer. Atkinson [11] studied the importance of loss systems when the arrival process is not well approximated by Poisson process. So it has been presented a discrete-time analysis of the $GI/G/2$ loss system which provides solving P_{loss} values. Also, Atkinson [12] investigated the $C_2/G/1$ queueing and the $C_2/G/1$ loss queue models. Assuming the c_X as the coefficient of variation of interarrival time and $\beta(s)$ as Laplace–Stieltjes transform of the service time distribution, it has been shown that the probability of delay and the probability of loss are both increasing in $\beta(s)$ for aforementioned two models when $c_X^2 < 1$. Samanta and Zhang [13] analyzed the discrete-time $GI/D\text{-}MSP/1$ queue with multiple vacations using the matrix-geometric method and Markov renewal theory approach. Liu et al. [14] analyzed the $M/M/1$ and $Geo/Geo/1$ queueing systems under single vacation policy from an economic perspective.

The assumption of identical servers is mostly invalid in real life. The Markovian queueing systems with heterogeneous servers have been widely studied in the literature. Gumbel [15] studied the $M/M/n$ queueing model with infinite waiting line and heterogeneous servers and provided the limit distribution of the number of customer. Singh [16] examined the Markovian queueing system with heterogeneous servers, computed the performance measures of the system, and compared the results with the homogeneous Markovian two-server model. Singh [17] obtained the steady-state probabilities, the mean number of customers waiting in the queue, and the mean waiting time in the system for the queuing model with infinite waiting line and three heterogeneous servers. Lin and Elsayed [18] developed a computer program to solve multi-channel Markovian ordered entry queuing system with heterogeneous servers and storage. Fakinos [19] presented a generalization of Erlang loss formula for the case of non-identical servers. It was proven by Nath and Enns [20] that the loss probability is minimal under the fastest service rule for the queuing model $M/M/n/0$ with heterogeneous servers. Elsayed [21] developed two computer programs to determine the optimal allocation of storage spaces among three heterogeneous servers in a finite source queuing system. Alpaslan and Shahbazov [22] proved that EW_q and EW take minimum values under the condition that $\mu_1 + \cdots + \mu_n = \mu$ for the $M/M/n$ model with heterogeneous servers when $\mu_1 = \mu_2 = \cdots \mu_n = c/n$. Kumar et al. [23] examined Markovian queuing model $M/M/2$ with heterogeneous servers and infinite waiting line while considering the fact that, catastrophes fitting the Poisson distribution with a rate of γ might take place. Alves et al. [24] derived upper bounds for the average number in queue L_q and the average waiting in queue W_q of heterogeneous multi-server Markovian queues, $M/M_i/c$. Choudhury and Deka [25] derived Laplace–Stieltjes transform of busy period distribution and waiting time distribution for the $M/G/1$ queue with two phases of heterogeneous service and unreliable server.

There are some studies on non-Markovian queuing systems with heterogeneity. Nawijn [26] considered the two-server queuing model with ordered entry and finite waiting rooms. In this model, it was assumed that the service time is exponential and the arrival process is deterministic. The overflow probability has been calculated for the defined queuing model by implementing the matrix solution. Alpaslan [27] obtained the distribution function of the stream of overflows for the $GI/M/2/0$ system with heterogeneous servers. Isguder and Uzunoglu-Kocer [28] minimized the loss probability according to the distribution of interarrival times for the $GI/M/3/0$ queueing model with heterogeneous servers and random entry. Furthermore,