Applied Mathematical Modelling 36 (2012) 6031–6049

Contents lists available at SciVerse ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

A spectral element method for solving the Pennes bioheat transfer equation by using triangular and quadrilateral elements

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ARTICLE INFO

Article history:
Received 3 October 2011
Received in revised form 1 January 2012
Accepted 4 January 2012
Available online 21 January 2012

Keywords:
Spectral element method
Pennes bioheat transfer equation
Lagrange polynomials
Dubiner polynomials
Fekete points
Triangular and quadrilateral elements

ABSTRACT

A spectral element method is developed for the numerical solution of the Pennes bioheat transfer equation which models the thermal behavior of the living tissue. For the one and two dimensional cases, the implementation of this method is completely explained. In the two dimensional case, both triangular and quadrilateral elements are investigated. Through test problems, the discretization error generated from this method is reported. In the triangular elements, the error is obtained when quadrature points coincide and do not coincide with nodal points. This method is employed to solve the equation in order to obtain the temperature of the skin layers, healthy tissue, and tissue that contains the tumor.

1. Introduction

The thermal behavior of living tissue can be mathematically described by the well-known Pennes bioheat transfer equation:

$$\rho c\frac{\partial u(\bar{x},t)}{\partial t}=\nabla\cdot[k\nabla u(\bar{x},t)]+\omega_b\rho_b c_b[u_a-u(\bar{x},t)]+Q_m+Q_r(\bar{x},t), \tag{1.1}$$

where $u(\bar{x},t)$ is the temperature of the living tissue at moment t and (vector) position $\bar{x}=[x_1,\ldots,x_d]$, d is the space dimension, ρ and c are the density and specific heat of the tissue, respectively, ρ_b and c_b are the density and specific heat of the blood, respectively, k denotes the thermal conductivity, ω_b stands for the blood perfusion rate, Q_m and Q_r are the volumetric heat source due to the metabolism and spatial heating respectively, and u_a is the arterial temperature [1,2]. The units of the symbols expressed in this equation are listed in Table 1. The Pennes bioheat transfer equation appeared in the pioneering work of Pennes [2].

Under certain conditions, the exact solution of the Pennes equation can be obtained [3,4]. To the best of our knowledge, in the general form the exact solution of this equation does not exist especially for the Pennes equation that the terms are changed during the domain. Therefore numerical methods are needed to solve this equation. In the test problems 3 and 4, we consider the Pennes equation that the terms are changed during the domain. In the test problem 3, the domain is the three skin layers i.e. epidermis, dermis, and subcutaneous. Since the values of the specific heat (c), blood perfusion rate (ω_b), thermal conductivity (k), and density (ρ) are not the same for these three layers, the terms of the equation i.e. $\rho c\frac{\partial u(\bar{x},t)}{\partial t}$, $\nabla\cdot[k\nabla u(\bar{x},t)]$, and $\omega_b\rho_b c_b[u_a-u(\bar{x},t)]$ are changed when the layer is altered. In the test problem 4, the domain

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doi:10.1016/j.apm.2012.01.018

Table 1
The unit of the symbols expressed in the Pennes equation.

Symbol	u	u_a	t	$\bar{x}$	ρ and ρ_b	c and c_b	k	ω_b	Q_m and Q_r
Unit	°C	°C	s	m	kg/m^3	J/kg °C	W/m °C	m^3/s/m^3	W/m^3

is the tissue that contains the tumor. Because the values of ω_b and the metabolic heat generation (Q_m) are different for the healthy tissue and tumor tissue, the terms of the equation i.e. $\omega_b\rho_b c_b[u_a - u(\bar{x},t)]$ and Q_m are changed during this domain.

Several numerical methods have been used for solving the Pennes bioheat transfer equation such as the finite difference method [5–10], finite element method [11–13], finite volume method [14,15], monte carlo method [16–18], boundary element method [19,20], dual reciprocity boundary element method (DRBEM) [21,22], multiple reciprocity boundary element method (MRBEM) [23], meshless method based on the radial basis functions and the method of fundamental solution (RBF-MFS) [24], cellular neural network (CNN) method [25] and FFT-based method [26]. The interested reader can see [27–31] for meshless methods for the numerical solution of partial differential equations.

The spectral element method was first presented by Patera [32] to find the numerical solution of the incompressible Navier–Stokes equations. This method is one of the high order methods because it performs the p-type refinement for decreasing the error and improving the accuracy.

The spectral element method inherits some properties of both the spectral methods [33,34] and the finite element methods. For infinitely smooth solutions, the spectral element method has the ability of achieving the high accuracy [35] similar to the spectral methods. But in the spectral element method against the spectral methods, approximating functions have the local support. Both the spectral element method and the h-version of the finite element method are based on the Galerkin approach that the approximating functions are the piecewise Lagrange polynomials while in these two methods the order of the approximation is different. Essential difference between the spectral element method and the p-version of the finite element method [36] is the nature of the local approximating functions. These functions in the spectral element method and the p-version finite element method are the Lagrange polynomials (cardinal functions) and the Legendre polynomials (orthogonal functions), respectively.

At first in this method, to take advantage of the tensor product, quadrilateral elements are used. Afterwards, due to the works of Dubiner [37], Taylor et al. [38], and Hesthaven [39], the possibility of obtaining local approximations defined over the triangular region is provided. Dubiner [37] obtained the orthogonal polynomials over the triangular region and Taylor et al. [38] and Hesthaven [39] derived the nodal points utilized for constructing the Lagrange polynomials over this region. Taylor et al. and Hesthaven proposed the Fekete points and the electrostatic points as the nodal points, respectively.

The spectral element method has been successfully applied to solve several problems such as the Black–Scholes equation [40], Klein–Gordon equation [41], elastic wave [42–44], acoustic wave [45,46], seismic wave [47,48], moving-boundary problems [49,50], the incompressible Navier–Stokes equations [51–53], Maxwell equations [54], shallow water equations [55,56], Helmholtz's equation [57], the P_N neutron transport equations [58] and the vector radiative transfer equation [59].

In this work, we apply the spectral element method to solve the one and two dimensional Pennes bioheat transfer equations. This method is considered in detail when triangular elements are used.

This paper is organized as follows. In Section 2, the spatial discretization of the one and two dimensional Pennes equations is presented via the spectral element method. In the two dimensional case both triangular and quadrilateral elements are investigated. For this purpose, the local approximating functions i.e. the Lagrange polynomials, the distribution of the nodal points and quadrature points, and the processes of differentiation and integration are investigated. In order to derive the Lagrange polynomials defined over the triangular region, Dubiner polynomials are explained. At the end, the system of ordinary differential equations obtained from the spatial discretization of the two dimensional Pennes equation with mixed Dirichlet and Neumann boundary conditions is presented. Section 3 includes four test problems. In problems 1 and 2 the accuracy of the spectral element method is considered. The error obtained from the numerical solution of the one and two dimensional Pennes equations is reported in problems 1 and 2, respectively. In problem 3 this method is applied for solving the Pennes equation under conditions that correspond to the three layers skin structure [1]. In problem 4 this method is employed to solve the Pennes equation under conditions that correspond to the structure of the healthy tissue and the tissue that contains the tumor [21,24]. Section 4 completes this paper with conclusions.

2. The spectral element method

In the spectral element method, the approximation of the order P for the function u satisfying the Pennes equation (1.1) is as follows:

$$u_P(\bar{x},t) = \sum_{i=1}^{N} u_P(\bar{x}_i,t)\phi_i(\bar{x}), \tag{2.1}$$

where N denotes the number of the grid points, $\{\phi_i\}_{i=1}^N$ are the global approximating functions of the order P, and $\{u_P(\bar{x}_i,t)\}_{i=1}^N$ are the global degrees of freedom (GDF). In this method the global approximating functions are the piecewise Lagrange polynomials and the global degrees of freedom are the values of the function u_P at the grid points $\{\bar{x}_i\}_{i=1}^N$. The global degrees of