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Polynomial and nonpolynomial spline methods for fractional sub-diffusion equations

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ABSTRACT

We consider one-dimensional fractional sub-diffusion equations on an unbounded domain. For a problem of this type for which an exact or approximate artificial boundary condition is available we reduce it to an initial-boundary value problem on a bounded domain. We then analyze the numerical solution of the problem by polynomial and nonpolynomial spline methods. The consistency and the Von Neumann stability analysis of these methods are also discussed. Numerical experiments clarify the effectiveness and order of accuracy of the proposed methods.

1. Introduction

In recent years, the derivation of solutions to fractional differential equations has become a hot topic in many fields of applied sciences and engineering. A large number of applied problems are formulated by fractional differential equations. In spite of many papers on numerical methods for fractional differential equations, there is still a lack of highly accurate numerical methods. The research on fractional order differential equations on unbounded domains is also of great importance. Using artificial boundary conditions (ABCs) is a widely used method for the solution of such problems [1].

There have been various applications of spline methods in the numerical solution of differential equations and particularly fractional differential equations, [2–5]. The author of [6] presented an implicit numerical method for fractional diffusion equation, discretizing the fractional derivative by spline and using the Crank–Nicolson discretization for time variable. The authors of [7] have developed a new nonpolynomial spline method for solving the second order hyperbolic equations and obtained better numerical results than those produced by some finite difference methods. The authors of [8] presented quadratic nonpolynomial spline approach for approximating the solution of a system of second-order boundary value problems associated with obstacle, unilateral, and contact problems and obtained approximations more accurate than those produced by collocation, finite difference and some standard polynomial spline methods. The authors of [9] have constructed cubic nonpolynomial spline functions for solving the non-linear Schrodinger equation. The authors in [4] have considered the numerical solution of the fractional boundary value problem (FBVP) by quadratic polynomial spline. In [3], the authors have used parametric spline functions for the solution of time fractional Burgers equation. In this paper, as a new investigation on the application and analysis of spline based methods for the solution of the problem in hand, we will consider polynomial and nonpolynomial splines and compare the obtained results with those obtained by finite difference method reported in [1].

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We consider the fractional differential equation on an unbounded domain expressed by

$$ {}_0^cD_t^{\gamma}u(x,t) - du_{xx}(x,t) = f(x,t), \quad (x,t) \in \Omega = [0,+\infty) \times [0,+\infty), \tag{1}$$

$$u(x,0) = \psi(x), \quad x \in [0,+\infty), \tag{2}$$

$$u(0,t) = \phi(t), \quad t > 0, \tag{3}$$

$$u(x,t) \to 0, \quad \text{when} \quad x \to +\infty,\ t > 0, \tag{4}$$

where ${}_0^cD_t^{\gamma}$ $(0 < \gamma < 1)$ is the Caputo fractional derivative of order γ defined by ${}_0^cD_t^{\gamma}f(t) = \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{f'(\tau)}{(t-\tau)^{\gamma}}\,d\tau$, and $d > 0$ is the diffusion coefficient, $f(x,t), \psi(x)$ and $\phi(t)$ are given functions and the support of $f(x,t)$ is the set $\Omega_1 = \{(x,t)|0 \leqslant x \leq 1, 0 \leqslant t < +\infty\}$ and the support of $\psi(x)$ is the interval $[0,1]$.

For computing the numerical solution of problem (1)–(4), we use an artificial boundary $\Gamma = \{(x,t)|x = 1,\ 0 \leqslant t < +\infty\}$. Then the domain Ω can be divided into the spatially bounded set Ω_1 and the unbounded set $\Omega_2 = \{(x,t)|1 \leqslant x < +\infty,\ 0 \leqslant t < +\infty\}$.

On the domain Ω_2, $f(x,t) \equiv 0$ and $\psi(x) \equiv 0$. We will consider the following boundary condition from [1]:

$$\frac{\partial u(1,t)}{\partial x} = \frac{-1}{\sqrt{d}}\,{}_0^cD_t^{\beta_1}u(1,t), \quad \beta_1 = \frac{\gamma}{2}. \tag{5}$$

The Eq. (5) represents the exact boundary condition of the problem (1)–(4) on the artificial boundary $\Gamma = \Omega_1 \cap \Omega_2 = \{(x,t)|x = 1, 0 \leqslant t < +\infty\}$. By the boundary condition (5), the original problem (1)–(4) on the unbounded domain Ω is then reduced to the following problem on the bounded domain Ω_1:

$${}_0^cD_t^{\gamma}u(x,t) - du_{xx}(x,t) = f(x,t), \quad (x,t) \in \Omega_1, \tag{6}$$

$$u(x,0) = \psi(x), \quad x \in [0,1], \tag{7}$$

$$u(0,t) = \phi(t), \quad t > 0, \tag{8}$$

$$\frac{\partial u(1,t)}{\partial x} = \frac{-1}{\sqrt{d}}\,{}_0^cD_t^{\beta 1}u(1,t), \quad t > 0 \tag{9}$$

with the parameters defined as before.

We will investigate the spline approximate solution of this problem in the rest of paper as follows. In Section 2, we use both polynomial and nonpolynomial spline functions for approximating the solution of the reduced problem (6)–(9). In Section 3, the theoretical analysis of local truncation error and the Von Neumann stability of the presented methods will be carried out. Finally, in Section 4, numerical results will be reported to illustrate the effectiveness and order of accuracy of the discussed methods.

2. Derivation of the methods

In this section, we present polynomial and nonpolynomial spline methods for solving the problem (6)–(9) in the finite interval $[0,T]$ (for the quadratic polynomial spline we use midknots). For the positive integers N and k, we take $h = \frac{1}{N}, \tau = \frac{T}{k}$ as the spatial stepsize and temporal stepsize, respectively. We use the notations $x_i = ih,\ (0 \leqslant i \leqslant N),\ t_j = j\tau,\ (0 \leqslant j \leqslant k),\ \Omega_h = \{x_i|0 \leqslant i \leqslant N\},\ \Omega_\tau = \{t_j|0 \leqslant j \leqslant k\},\ u_i^j = u(x_i,t_j)$ and $z_i^j = z(x_i,t_j)$.

The bounded domain $[0,1] \times [0,T]$ is then covered by $\Omega_h \times \Omega_\tau$. For any mesh function $w = \{w_i^j|0 \leqslant i \leqslant N, 0 \leqslant j \leqslant k\}$defined on $\Omega_h \times \Omega_\tau$, the following notations are introduced:

$$w_{i+\frac{1}{2}}^j = \frac{1}{2}\left(w_{i+1}^j + w_i^j\right), \quad w_{i-\frac{1}{2}}^j = \frac{1}{2}\left(w_i^j + w_{i-1}^j\right), \quad w_{i-\frac{3}{2}}^j = \frac{1}{2}\left(w_{i-1}^j + w_{i-2}^j\right).$$

Let z_i^j be an approximate value of $u(x_i,t_j)$, obtained by the spline function $p_i(x,t_j)$ passing through the points (x_i,z_i^j) and (x_{i+1},z_{i+1}^j).

2.1. Polynomial spline forms

2.1.1. Quadratic polynomial spline

Let us consider the quadratic polynomial spline $p_i(x,t)$ in the form [4,10]

$$p_i(x,t_j) = a_i(t_j)(x-x_i)^2 + b_i(t_j)(x-x_i) + c_i(t_j), \quad i = 0,1,2,\ldots,N-1. \tag{10}$$