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Numerical solution of fractional diffusion-wave equation based on fractional multistep method

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ABSTRACT

This paper is devoted to application of fractional multistep method in the numerical solution of fractional diffusion-wave equation. By transforming the diffusion-wave equation into an equivalent integro-differential equation and applying Lubich's fractional multistep method of second order we obtain a scheme of order $O(\tau^\alpha + h^2)$ for $1 \leqslant \alpha \leqslant 1.71832$ where α is the order of temporal derivative and τ and h denote temporal and spatial stepsizes. The solvability, convergence and stability properties of the algorithm are investigated and numerical experiment is carried out to verify the feasibility of the scheme.

1. Introduction

Fractional calculus has attracted considerable attention since the first time it was introduced as the counterpart of integral order calculus. A large amount of theoretical work has been done in this field and we can refer to the encyclopedic book written by Samko et al. [1] for most of the significant theoretical achievements. In recent decades people have found fractional model more appropriate to describe various phenomenons with memorial, retarding, hereditary or non-Markovian (when associated with stochastic process) characteristics. There are rich literatures on the applications of fractional models in different fields in science and engineering such as viscoelasticity [2], seepage [3], hydrology [4–6] and even finance [7–10]. As we all notice well, differential equation is a tremendously important model to describe several processes. When the traditional integer order differential equation fails to accurately describe the process under consideration we resort to the fractional order analogue. Naturally the time fractional differential equation can be derived by an substitution of time derivative of fractional order $\alpha > 0$ for the first or second order time derivatives correspondingly in the classical diffusion or wave equation. We refer to the derived equation as time fractional diffusion equation when $0 < \alpha < 1$ (considered by Gorentflo et al. [11] when solving continuous time random walk problems) and as time fractional diffusion-wave equation when $1 < \alpha < 2$ which is accepted as an interpolation of the diffusion and wave equation.

Recently, many published papers are devoted to the numerical solution of time fractional diffusion equation. Yuste and Acedo [12,13] construct a forward-Euler scheme and weighted average finite difference scheme for time fractional diffusion equation and give stability analysis. However, they only give the local truncation error and do not derive the convergence order of their approaches. Liu et al. [14] present a first order approximation in temporal and spatial direction for this equation and analyze the stability property of the scheme. Zhuang and Liu [15] derive a $O(\tau + h^2)$ scheme for the fractional

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sub-diffusion equation and analyze the convergence and stability properties. Chen et al. [16] consider the fractional diffusion equation with Kansa method in which the MultiQuadrics and thin plate spline are chosen to be the radial basis function. Fu and Chen [17] propose boundary particle meshless method for Laplace transformed time fractional diffusion equation with either Dirichlet or mixed boundary condition. Xu et al. [18,19] consider the time fractional equation with spectral method. In [18] they discretize both temporal and spatial derivatives using spectral method. In [19] they discretize temporal derivative by finite difference method while spatial derivative by Legendre spectral method. Moreover, stability and convergence of the method are analyzed, and a scheme with $2-\alpha$ order accuracy in temporal direction and spectral accuracy in spatial direction is obtained. It should be pointed out that applications of spectral method suffer so much from the requirement of smoothness of the analytic solution to the equation under consideration.

As the time fractional diffusion equation, some investigations are also carried out on the numerical approximation of time fractional diffusion-wave equation. Sun [20] construct a $O(\tau^{3-\alpha}+h^2)$ (refer to abstract for the symbols) full discrete difference scheme for the diffusion-wave equation by introducing two intermediate variables and analyze the stability and convergence properties using energy method. Du and Cao [21] treat the same equation with compact difference method and obtain a scheme of order $O\left(\tau^{3-\alpha}+h^4\right)$. Tadjeran et al. [22] present a second-order approximation for the fractional diffusion equation with Riemman–Liouville derivative in spatial direction based on Crank–Nicholson method. However, their second order approach is based on Richardson extrapolation. In this article, we present a $O(\tau^{\alpha}+h^2)$ scheme for the fractional diffusion-wave equation with Caputo temporal derivative using Lubich's [23] fractional multistep method (FMM). Obviously our scheme has a higher convergence order when $\alpha>1.5$ than Sun's in the temporal direction. The FMM first introduced into the quadrature of fractional integrals by Lubich [23] in 1986 provides us a new way to derive high order difference scheme. But applications of FMM in the numerical solution of fractional differential equation are rarely reported yet. Lin and Liu [24] consider the nonlinear fractional ordinary differential equation with this method and give a detailed analysis of the convergence and stability. However, to the extent of our knowledge, application of this method in the solution of time fractional diffusion-wave equation is still to be developed greatly. Not as the well known $L1$, $L2$ and block-by-block [25] approximation of fractional order derivatives, the application of FMM brings in convolution quadrature weights which are not expressed explicitly but determined by a generation function (consequently by a recurrence equation) and makes the theoretical analysis quite difficult. This paper can be considered as an exploration of the application of FMM in the numerical solution of fractional partial differential equation. Herein we first transform the original equation into an equivalent integro-differential equation as we did in [26] and then use second order FMM to discretize the integral term. We discuss the mathematical aspects of the quadrature convolution weights and investigate the convergence order and stability of the proposed scheme using energy method and finally verify the theoretical result by numerical experiment.

The remainder of this paper is organized as follows. In Section 2 we give an equivalent integro-differential equation of the fractional diffusion-wave equation and construct the difference scheme. In Section 3 we discuss the properties of the convolution weights and analyze the solvability, convergence order and stability of the proposed method. Numerical experiment is presented in Section 4. In the final section we draw some conclusions.

2. Derivation of the difference scheme

Consider the one-dimensional fractional diffusion-wave equation [27]

$$\frac{1}{c}\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{\partial^2 u}{\partial x^2}+\frac{1}{K}f(x,t),\quad 0\leqslant x\leqslant L,\ t>0, \tag{1}$$

with the initial conditions

$$u(x,0)=\varphi(x),\quad \frac{\partial u(x,0)}{\partial t}=\psi(x),\ 0\leqslant x\leqslant L,$$

and the boundary conditions

$$u(0,t)=0,\quad u(L,t)=0,\ t>0,$$

where c and K are constants and $f(x,t)$ denotes the source term, x and t are space and time variables and

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}\frac{\partial^2 u(x,s)}{\partial s^2}ds,\quad (1<\alpha<2) \tag{2}$$

is Caputo derivative with respect to t with Γ being the gamma function. In fact, we can rewrite the Caputo derivative as following

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\frac{1}{\Gamma(1-(\alpha-1))}\int_0^t (t-s)^{-(\alpha-1)}\frac{\partial}{\partial s}\frac{\partial u(x,s)}{\partial s}ds=\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}\frac{\partial}{\partial t}u(x,t). \tag{3}$$

Using formula (3), we have.