Applied Mathematical Modelling 37 (2013) 9617–9629

Contents lists available at SciVerse ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

On homotopy analysis method applied to linear optimal control problems

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A R T I C L E I N F O

Article history:
Received 15 January 2012
Received in revised form 1 March 2013
Accepted 13 May 2013
Available online 2 June 2013

Keywords:
Homotopy analysis method
Optimal control problems
Pontryagin's maximum principle
Hamiltonian system

A B S T R A C T

In this paper, the homotopy analysis method (HAM) is employed to solve the linear optimal control problems (OCPs), which have a quadratic performance index. The study examines the application of the homotopy analysis method in obtaining the solution of equations that have previously been obtained using the Pontryagin's maximum principle (PMP). The HAM approach is also applied in obtaining the solution of the matrix Riccati equation. Numerical results are presented for several test examples involving scalar and 2nd-order systems to demonstrate the applicability and efficiency of the method.

1. Introduction

Optimal control is one particular branch of modern control theory which has a wide range of practical applications not only in all areas of physics but also in economy, aerospace, chemical engineering, robotic, etc. (see [1–4]).

Linear optimal control is a special sort of optimal control. The plant that is controlled is assumed linear, and the controller, the device which generates the optimal control, is constrained to be linear. Many computational techniques have been developed for solving OCPs based on Pontryagin's maximum principle [5] or Hamilton–Jacobi–Bellman equation[6]. Such as successive approximation approach (SAA) [4], the modal series method [7], homotopy perturbation method (HPM) [8–10] and differential transform method (DTM) [11].

The homotopy analysis method (HAM) is a powerful analytical tool for solving nonlinear problems. The HAM was first proposed by Liao [12], by employing the basic ideas of homotopy in topology to produce an analytical method for solving various nonlinear problems. This technique provides a simple way to ensure the convergence of the solution series, so that we can always get accurate enough approximations. Furthermore, the homotopy analysis method logically contains the non-perturbation methods such as Lyapunov's artificial small parameter method [13], Adomian's decomposition method [14] and homotopy perturbation method [15]. Since Liao's book [16] about the homotopy analysis method was published in 2003, more and more researchers have been successfully applying this method to various nonlinear problems in science and engineering, such as the viscous flows of non-Newtonian fluids [17], the KdV-type equations [18] and so on. This shows the great potential of the HAM for strongly nonlinear problems in science and engineering.

In this paper, we consider the following linear optimal control problem

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http://dx.doi.org/10.1016/j.apm.2013.05.009

$$\begin{aligned}
&\min\ J = \frac{1}{2}x(t_f)^T Sx(t_f) + \frac{1}{2}\int_{t_0}^{t_f}(x^T Px + 2x^T Qu + u^T Ru)dt,\\
&s.t.\ \dot{x} = Ax(t) + Bu(t),\\
&x(t_0) = x_0,
\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$. The control $u(t)$ is an admissible control if it is piecewise continuous in t for each t. Its values belong to a given closed subset U of $\mathbb{R}^+$. The input $u(t)$ is derived by minimizing the quadratic performance index J, where $S \in \mathbb{R}^{n\times n},\ P \in \mathbb{R}^{n\times n}$ is positive semi-definite matrices, $R \in \mathbb{R}^{m\times m}$ is positive definite matrix and $Q \in \mathbb{R}^{n\times m}$.

The contribution of our paper is to apply the HAM for solving the linear OCPs. The application of Pontryagin's maximum principle (PMP) to the linear OCPs results in a system of two point boundary value problem (TPBVP). As we will point out in Section 2, we can achieve the feedback optimal control law, by using PMP.

The paper has been organized as follows. Section 2, describes the solution guidelines for linear optimal control system (1). Section 3, presentation Steady-state Riccati equation. In Section 4, HAM is applied for solving optimal control problem. Finally, conclusions are given in the last section.

2. Solution guidelines for linear optimal control system

In this section, we present an efficient technique, based on the HAM, for solving linear optimal control problem (1). According to the PMP, we have

$$\begin{cases} \dot{x} = [A - BR^{-1}Q^T]x - BR^{-1}B^T\lambda, \\ \dot{\lambda} = [-P + QR^{-1}Q^T]x + [QR^{-1}B^T - A^T]\lambda, \end{cases} \tag{2}$$

with the condition $x(t_0) = x_0$. Also, the optimal control law is

$$u^* = -R^{-1}Q^Tx - R^{-1}B^T\lambda, \tag{3}$$

since $x(t_f)$ is indeterminate, then

$$\lambda(t_f) = Sx(t_f). \tag{4}$$

system (2) is linear, therefore, we can write solution of system (2) in following form

$$\begin{pmatrix} x \\ \lambda \end{pmatrix} = \phi(t,t_f)\begin{pmatrix} x(t_f) \\ \lambda(t_f) \end{pmatrix}, \tag{5}$$

where ϕ is an $2n \times 2n$ matrix. Also

$$\phi(t,t_f) = \begin{pmatrix} F(t,t_f) & G(t,t_f) \\ L(t,t_f) & M(t,t_f) \end{pmatrix}, \tag{6}$$

where $F,\ G,\ L$ and M are $n \times n$ matrices. Now by applying the terminal condition $\lambda(t_f) = Sx(t_f)$ and since $F + GS$ is invertible, we have

$$\lambda(t) = (L + MS)(F + GS)^{-1}x(t), \tag{7}$$

or

$$\lambda(t) = Y(t,t_f)x(t), \tag{8}$$

where

$$Y(t,t_f) = \big(L(t,t_f) + M(t,t_f)S\big)\big(F(t,t_f) + G(t,t_f)S\big)^{-1}. \tag{9}$$

Differentiating Eq. (8) with respect to t, leads to

$$\dot{\lambda}(t) = \dot{Y}(t,t_f)x(t) + Y(t,t_f)\dot{x}(t), \tag{10}$$

Substituting $\dot{x},\ \dot{\lambda}$ and λ from Eqs. (2) and (8) into (10), we have

$$(-\dot{Y} + (YB + Q)R^{-1}(B^TY + Q^T) - YA - A^TY - P)x(t) = 0.$$

Since the above equation must hold for all nonzero $x(t)$, then $Y(t,t_f)$ must satisfy the matrix Riccati equation,

$$\dot{Y} = (YB + Q)R^{-1}(B^TY + Q^T) - YA - A^TY - P. \tag{11}$$

By considering Eqs. (3) and (8), we can see that the optimal control law as

$$u^*(t) = -R^{-1}Q^Tx - R^{-1}B^TY(t,t_f)x(t). \tag{12}$$