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Integration of barotropic vorticity equation over spherical geodesic grid using multilevel adaptive wavelet collocation method

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ABSTRACT

In this paper, we present the multilevel adaptive wavelet collocation method for solving non-divergent barotropic vorticity equation over spherical geodesic grid. This method is based on multi-dimensional second generation wavelet over a spherical geodesic grid. The method is more useful in capturing, identifying, and analyzing local structure [1] than any other traditional methods (i.e. finite difference, spectral method), because those methods are either full or partial miss important phenomena such as trends, breakdown points, discontinuities in higher derivatives of the solution. Wavelet decomposition is used for interpolation and adaptive grid refinement on different levels.

1. Introduction

The barotropic vorticity equation model is an important equation in the research of the atmospheric sciences which describes the evolution of the vorticity of a fluid element as it moves around. It is a simplification of conservation law of momentum for inviscid and incompressible fluid. For the theoretical investigations of the evolution of vortices, atmospheric researchers are using the barotropic assumption, as there is no vertical component, i.e., single-layered fluid. Moreover, barotropic model is useful for modeling the movement of tropical cyclones [2–4] and the interaction of two vortices in close proximity to one another [5]. The barotropic assumption has also been used to model global wave patterns in the middle troposphere [6,7]. But sometime to find analytic solutions of these type of problems are either not known or very difficult to develop. Therefore, many scientists pay attention to the research of numerical methods of the equation [8–10].

Since atmospheric blockings are approximately stationary and relatively long-lived phenomena, so that one might attempt to describe them in term of stationary solution of barotropic vorticity equation [11,12]. However the baropropic vorticity equation on a sphere has known several stationary or longitudinally propagating solution, such as exact solution of Rossby–Hauritz wave [13] and modons [14,15]. Examples of numerical solution obtained for Rossby–Hauritz wave [16] and modons [17,14]. This solution of modon and Rossby–Hauritz is to be presented here with less computational cost and clearly indicating the region of sharp gradient.

The theory and application of wavelets has become an active area of research in different fields, including electrical engineering (signal processing, data compression), mathematical analysis (harmonic analysis, operator theory), and physics (fractals, quantum field theory). Moreover, it also applied to seismic signal studies in geophysics; and applications in turbulence studies in the atmospheric sciences. Basically application of signal analysis in atmosphere sciences has two main directions as followed: the singularity and the variance analysis.

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The current wavelet method can be classified in different ways depending on the above applications whether it take full or partial advantage of wavelet analysis (i.e. multiresolution properties, wavelet compression, the detection of localized structures and subsequent use for grid adaptation, fast wavelet transform, wavelet-based interpolation, and active error control) [18]. But still now it's application for solving partial differential equations (PDEs) on general manifold is in infancy stage. A new adaptive second generation wavelet collocation method for solving PDEs on sphere has recently been developed in [19]. The adaptive wavelet collocation method is most appropriate for solving nonlinear PDEs with general boundary conditions. This approach combines the adaptivity and error control of the adaptive wavelet method with the flexibility of collocation. It has been verified by many authors in [20,21,1,22,23] over the flat geometry and [19,24] on sphere. Therefore, the aim of this paper is to apply multilevel adaptive wavelet collocation method (MAWCM) for solving useful barotropic vorticity equation on the sphere. Since wavelets are localized in both space and scale, we can clearly analyze local structure of any kind. Furthermore the computational cost of the MAWCM is $O(\mathcal{N})$ which is independent of the dimension of the problem, where $\mathcal{N}$ is the total number of collocation points.

The paper is organized as follows, the brief introduction about second generation wavelet is given in Section 2. In Section 3, we are discussing MAWCM to solve PDEs on the sphere. Moreover in section 4 we describe clearly how operators (Jacobian operator and Laplace–Beltrami operator) are calculated on an adaptive grid. In Section 5, the numerical experiment of two test cases are given. The conclusion is outlined in Section 6.

2. Spherical wavelets

Some of the first non-trivial wavelets that have been developed are the Daubechies wavelet [25], Coiflets [25,26], Meyer wavelet [27] and Morlet wavelet [25,28]. These, and most other wavelets developed in the 1980s, are first generation wavelets whose construction requires the Fourier transform and whose basis functions have to be dilation and translation of single function (mother wavelet). However, these wavelets were limited to flat geometries. The work by Swelden [29] overcome these restrictions and led to the second generation wavelets on general manifold.

The construction of spherical wavelet (second generation wavelet) in [30] relies on recursive partitioning of the sphere into spherical triangles. This is done staring from a platonic solid whose faces are spherical triangles. Here we consider the icosahedral subdivision for which $\mathcal{K}^j = 10 \times 4^j + 2$ at subdivision level j. Let $\mathcal{S}$ be a triangulation of the sphere S and denote the set of all vertices obtained after subdivisions with $\mathcal{S}^j = \{p_k^j \in S | k \in \mathcal{K}^j\}$, where $\mathcal{K}^j$ is an index set. Now the original platonic solid icosahedral $\mathcal{S}^0$ contains only 12 vertices and the $\mathcal{S}^1$ contains those vertices and all new vertices on the edge midpoints. Since $\mathcal{S}^j \subset \mathcal{S}^{j+1}$ we also let $\mathcal{K}^j \subset \mathcal{K}^{j+1}$. Let $\mathcal{M}^j = \mathcal{K}^{j+1}/\mathcal{K}^j$ be the indices of the vertices added when going from level j to $j+1$.

A second generation multi resolution analysis (MRA) [29] of the sphere provides a sequence $\mathcal{V}^j \subset L_2(S)$ with $j \geqslant 0$; and the sphere $S = \{p = (p_x, p_y, p_z) \in \mathbb{R}^3 : \|p\| = r\}$, where r is the radius of the sphere:

- $\mathcal{V}^j \subset \mathcal{V}^{j+1}$,
- $\bigcup_{j \geqslant 0} \mathcal{V}^j$ is dense in $L_2(S)$,
- each $\mathcal{V}^j$ has a Riesz basis of scaling functions $\{\phi_k^j | k \in \mathcal{K}^j\}$.

Since $\phi_k^j \in \mathcal{V}^j \subset \mathcal{V}^{j+1}$, for every scaling function ϕ_k^j filter coefficients $h_{k,l}^j$ exists such that

$$\phi_k^j = \sum_{l \in \mathcal{K}^{j+1}} h_{k,l}^j \phi_l^{j+1}. \tag{1}$$

Note that the filter coefficients $h_{k,l}^j$ can be different for every $k \in \mathcal{K}^j$ at a given level $j \geqslant 0$. Therefore each scaling function satisfies a different refinement relation. Each MRA is accompanied by a dual MRA consisting of nested spaces $\tilde{\mathcal{V}}^j$ with bases by the dual scaling functions $\tilde{\phi}_k^j$, which are biorthogonal to the scaling functions:

$$\langle \phi_k^j, \tilde{\phi}_{\acute{k}}^j \rangle = \delta_{k,\acute{k}}, \quad \text{for } k, \acute{k} \in \mathcal{K}^j, \tag{2}$$

where $\langle f, g \rangle = \iint_s fgdw$ is the inner product on the sphere. The dual scaling functions satisfy refinement relations with coefficients $\{\tilde{h}_{k,l}^j\}$. The surface plot of scaling function and it's cross cut along maximum and minimum are plotted in Fig. 1.

One most important thing when you are going to build MRA to construction of wavelets. They encode the difference between two successive levels of representation, that is there from Riesz basis for the space $\mathcal{W}$, which is complement of $\mathcal{V}^j$ in $\mathcal{V}^{j+1}$ (i.e. $\mathcal{V}^{j+1} = \mathcal{V}^j \oplus \mathcal{W}^j$). The construction of the wavelets form a Riesz basis for $L^2(S)$ and allow a function to be represented by its wavelet coefficients. Since $\mathcal{W}^j \subset \mathcal{V}^{j+1}$, we can write

$$\psi_k^j = \sum_{l \in \mathcal{K}^{j+1}} g_{k,l}^j \phi_l^{j+1}, \tag{3}$$

and the spherical wavelets ψ_m^j have $\tilde{d}$ vanishing moments, if $\tilde{d}$ is the independent polynomials $P_i, \ 0 \leqslant i \leqslant \tilde{d}$ exist such that

$$\langle \psi_m^j, P_i \rangle = 0 \quad \forall j \geqslant 0, \ m \in \mathcal{M}^j, \tag{4}$$