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Numerical solutions to initial and boundary value problems for linear fractional partial differential equations

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ABSTRACT

In this article, Haar wavelets have been employed to obtain solutions of boundary value problems for linear fractional partial differential equations. The differential equations are reduced to Sylvester matrix equations. The algorithm is novel in the sense that it effectively incorporates the aperiodic boundary conditions. Several examples with numerical simulations are provided to illustrate the simplicity and effectiveness of the method.

1. Introduction

In recent years fractional order partial differential equations have attracted the attentions of many researchers and are becoming increasingly popular due to their practical applications in various fields of science and engineering [1]. It is due to the fact that mathematical models based on factional order derivatives either with respect to time or space or both are more realistic and efficiently describe variety of natural phenomena. Solutions of integer order partial differential equations describe the future states of some physical processes from the knowledge of their present states and are independent of their past history. However, for the description of memory and hereditary properties of various materials and processes, where the past states exert their influence in a significant way on the future states, integer order models are not sufficient to handle the situation. In such cases, and many more, fractional order partial differential equations provide excellent tools for the description of the systems.

Further, it is worthwhile to mention that explicit analytic solutions of fractional order partial differential equations are rarely available in the literature; the need to exploit some efficient and reliable numerical scheme is a problem of fundamental interest. In the literature, a number of powerful numerical methods have been proposed for obtaining approximate solutions to fractional order partial differential equations. To name a few, Odibat [2] used a modification of the Adomian decomposition method for solving fractional diffusion–wave equations. Abdulaziz et al. [3] obtained approximate analytical solutions for fractional order KdV equations with the homotopy perturbation method. In [4–6], the generalized differential transform method is used to solve linear partial differential equations of fractional order and time-fractional reaction–diffusion equations. Moaddy [7] et al. have developed a finite difference scheme to solve the linear partial differential equations with time- and space-fractional derivatives.

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Recently, orthogonal wavelet bases are becoming more popular for numerical solutions of partial differential equations. It is because of their excellent properties such as, ability to detect singularities, orthogonality, flexibility to represent a function at different levels of resolution, and compact support. In recent years, there has been a growing interest in developing wavelet based numerical algorithms for solutions of fractional order partial differential equations. Among them, the Haar wavelet method is the simplest and is easy to use. The discontinuity of the Haar wavelets prevents one form using them directly for solutions of partial differential equations. To overcome this difficulty, one has to approximate the highest order derivative of the unknown function, either with respect to space or time, by Haar wavelets series and integrate it to approximate other terms in the equation.

Such an approach was initially suggested by Chen and Hsiao in [8] for system analysis. Since then Haar wavelets have been successfully applied for the solutions of ordinary and partial differential equations, integral equations and integro-differential equations [9–13]. Recently, there has been some considerable developments in the application of wavelet basis to solve differential equations of fractional order [14–18].

In this work we focus on linear fractional partial differential equations. In particular, we will focus on the class of fractional partial differential equations

$$\frac{\partial^{\gamma} u(x,t)}{\partial t^{\gamma}} - a(x)\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + b(x)\frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} + d(x)u(x,t) = f(t,x), \quad 0 < \gamma \leqslant 2, \quad 1 < \alpha \leqslant 2, \quad 0 < \beta \leqslant 1,$$

where the partial fractional derivatives are taken in the Caputo sense. For $0 < \alpha < 1$; $1 < \beta < 2$, $a(x) > 0$, $b(x) > 0$, $d(x) \geqslant 0$ continuous on $[0, L]$, $L > 0$ and $f(x,t)$ continuous on $[0, L] \times [0, L]$, the problem reduces to fractional convection–diffusion equations. Zhang [19] proposed an implicit unconditional stable difference scheme for the numerical solutions of this kind of convection–diffusion equation satisfying initial condition $u(x, 0) = 0$ and Dirichlet boundary conditions $u(0) = u(1) = 0$. For constants $a, b > 0$, $d = 0$ and with $f(x,t) = 0$, the problem reduces to space–time fractional advection–dispersion equation, which have been considered by Ram Pandey [20] with non-homogenous initial conditions. Also, some other researchers have worked on the fractional convection–diffusion and advection–dispersion equations, including [21–27]. Furthermore, in [28] the problem is solved by the Haar wavelet method. The difficulty one encounters in the implementation of wavelet based methods is the enforcement of non-periodic boundary conditions. One of the advantage of present work is that it efficiently incorporates the aperiodic boundary conditions.

The algorithm considered in this work is based on operational matrices of fractional order integration and it effectively generalizes the method used in [29] for fractional partial differential equations.

2. Fractional derivative and integral

For convenience, this section summarizes some concepts, definitions and basic results from fractional calculus, which are useful for the further developments in this paper.

Definition 2.1 (*[30,31]*). Let $\alpha > 0$, $n = \lceil \alpha \rceil$ and $u(x,t) \in C^n([0,1] \times [0,1])$. Then the partial Caputo fractional derivative of $u(x,t)$ with respect to t is defined as

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} u(x,t) = \begin{cases} \mathcal{I}_t^{n-\alpha} \frac{\partial^n}{\partial t^n} u(x,t), & n-1 < \alpha \leqslant n, \\ \frac{\partial^n}{\partial t^n} u(x,t), & \alpha = n \in \mathbb{N}, \end{cases}$$

where $\mathcal{I}_t^{\alpha}$ is the Riemann–Liouville fractional integral, given as

$$\mathcal{I}_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \mathcal{I}_t^0 f(t) := f(t). \tag{2.1}$$

We review few basic properties of fractional differential and integral operators that are needed for this work. For details, we refer to [31].

(P_1) $\mathcal{I}_t^{\alpha} \mathcal{I}_t^{\beta} f(t) = \mathcal{I}_t^{\alpha+\beta} f(t) = \mathcal{I}_t^{\beta} \mathcal{I}_t^{\alpha} f(t)$,
(P_2) $\frac{\partial^{\beta}}{\partial t^{\beta}} \mathcal{I}_t^{\alpha} u(x,t) = \mathcal{I}_t^{\alpha-\beta} u(x,t)$,
(P_3) $\mathcal{I}_t^{\alpha} \frac{\partial^{\alpha}}{\partial t^{\alpha}} u(x,t) = u(x,t) - \sum_{j=0}^{n-1} \frac{t^j}{j!} \frac{\partial^j u(x,t)|_{t=0}}{\partial t^j} = u(x,t) + \sum_{j=0}^{n-1} \mu_j(x) t^j$, where, $\mu_j(x) = -\frac{1}{j!} \frac{\partial^j u(x,t)|_{t=0}}{\partial t^j}$.

The Caputo fractional derivative of order $\alpha > 0$ for $g(t) := t^{\beta}$, is given as [31]

$$\mathcal{D}^{\alpha} g(t) = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & \text{if } \beta \in \mathbb{N}, \text{ and } \beta \geqslant m \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > m-1, \\ 0, & \text{if } \beta \in \{0, 1, 2, \ldots, m-1\}. \end{cases} \tag{2.2}$$

For a function of one variable, we use the notation $\mathcal{D}^{\alpha}$ instead of $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ for the Caputo fractional derivative. Furthermore, the Caputo fractional derivatives of sine and cosine functions are:

$$\mathcal{D}^{\alpha} \sin(t) = t^{1-\alpha} E_{2,2-\alpha}(-t^2), \quad \mathcal{D}^{\alpha} \cos(t) = t^{-\alpha} E_{2,1-\alpha}(-t^2),$$