Applied Mathematical Modelling 37 (2013) 5498–5510

Contents lists available at SciVerse ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Fractional-order Legendre functions for solving fractional-order differential equations

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ARTICLE INFO

Article history:
Received 22 January 2012
Received in revised form 15 September 2012
Accepted 15 October 2012
Available online 9 November 2012

Keywords:
Fractional-order Legendre functions
Fractional-order differential equations
Operational matrix
Tau method

ABSTRACT

In this article, a general formulation for the fractional-order Legendre functions (FLFs) is constructed to obtain the solution of the fractional-order differential equations. Fractional calculus has been used to model physical and engineering processes that are found to be best described by fractional differential equations. Therefore, an efficient and reliable technique for the solution of them is too important. For the concept of fractional derivative we will adopt Caputo's definition by using Riemann–Liouville fractional integral operator. Our main aim is to generalize the new orthogonal functions based on Legendre polynomials to the fractional calculus. Also a general formulation for FLFs fractional derivatives and product operational matrices is driven. These matrices together with the Tau method are then utilized to reduce the solution of this problem to the solution of a system of algebraic equations. The method is applied to solve linear and nonlinear fractional differential equations. Illustrative examples are included to demonstrate the validity and applicability of the presented technique.

1. Introduction

Fractional derivatives have a long mathematical history, but they were not used in physics for many years. One possible explanation of such unpopularity could be that there are multiple nonequivalent definitions of fractional derivatives [1]. Another difficulty is that fractional derivatives have no evident geometrical interpretation because of their nonlocal character [2]. However, during the last 10 years fractional calculus starts to attract much more attention of physicists and mathematicians. It was found that various, especially interdisciplinary applications can be elegantly modeled with the help of the fractional derivatives. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives [3], and the fluid-dynamic models with fractional derivatives [4,5] can eliminate the deficiency arising from the assumption of continuum traffic flow. Based on experimental data fractional partial differential equations for seepage flow in porous media are suggested in [6], and differential equations with fractional order have recently proved to be valuable tools to the modeling of many physical phenomena [1,7]. A review of some applications of fractional derivatives in continuum and statistical mechanics is given by Mainardi [8]. The analytic results on the existence and uniqueness of solutions to the fractional differential equations have been investigated by many authors [1,9]. During the last decades, several methods have been used to solve fractional differential equations, fractional partial differential equations, fractional integro-differential equations and dynamic systems containing fractional derivatives, such as Adomian's decomposition method [10–14], homotopy analysis method [15–18], and other methods [19–24].

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0307-904X/$ - see front matter © 2012 Published by Elsevier Inc.
http://dx.doi.org/10.1016/j.apm.2012.10.026

Orthogonal functions have received considerable attention in dealing with various problems. The main characteristic behind the approach using this technique is that it reduces these problems to those of solving a system of algebraic equations thus greatly simplifying the problem. In the present paper, we intend to extend the application of the new orthogonal function based on Legendre polynomial to solve fractional differential equations. Our main aim is to generalize the fractional-order Legendre function operational and product matrices to fractional calculus. It is worthy to mention here that, the method based on using the operational matrix of an orthogonal function for solving differential equations is computer oriented.

The remainder of this paper is organized as follows: we begin by introducing some necessary definitions and mathematical preliminaries of the fractional calculus theory. In Section 3, the fractional-order Legendre functions and their properties is obtained. Section 4 is devoted to apply the FLFs operational matrices of fractional derivative and product to obtain the solution of fractional differential equation. In Section 5, the proposed method is applied to several examples. Also a conclusion is given in the last Section.

2. Preliminaries and notations

In this section, we give some basic definitions and properties of fractional calculus theory which are further used in this article.

Definition 1. A real function $f(x),\ x>0$ is said to be in space $C_\mu,\ \mu\in\mathbb{R}$ if there exists a real number $p>\mu$, such that $f(t)=t^pf_1(t)$, where $f_1(t)\in C(0,\infty)$, and it is said to be in the space C_μ^n if and only if $f^n\in C_\mu,\ n\in\mathbb{N}$.

Definition 2. The Riemann–Liouville fractional integral operator of order $\alpha>0$, of a function $f\in C_\mu,\ \mu\geqslant -1$, is defined as

$$I^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds,\quad \alpha>0,$$
$$I^0f(t)=f(t).$$

Definition 3. The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^\alpha f(t)=I^{m-\alpha}D^mf(t),$$

for $m-1<\alpha\leqslant m,\ m\in\mathbb{N},\ t>0$ and $f\in C_{-1}^m$.

Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative. Some properties of the operator D^α, which are needed here, are as follows

For $f\in C_\mu,\ \mu\geqslant -1,\ \alpha,\beta\geqslant 0,\ \gamma\geqslant -1,\ \mathbb{N}_0=\{0,1,2,\ldots\}$ and constant C

(1) $D^\alpha D^\beta f(t)=D^{\alpha+\beta}f(t)$,

(2) $D^\alpha C=0$,

(3) $$D^\alpha t^\gamma=\begin{cases}0, & \gamma\in\mathbb{N}_0\quad\text{and}\quad \gamma<\alpha,\\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}t^{\gamma-\alpha}, & \text{Otherwise}.\end{cases}\tag{1}$$

Similar to the integer-order derivative, the Caputo fractional derivative is a linear operation

$$D^\alpha\left(\sum_{i=1}^n c_if_i(t)\right)=\sum_{i=1}^n c_iD^\alpha f_i(t),\tag{2}$$

where $\{c_i\}_{i=1}^n$ are constants.

Now we define a generalization of Taylor's formula involves Caputo fractional derivatives which is introduced by Odibat and Momani [25].

Definition 4 (*Generalized Taylors formula*). Suppose that $D^{k\alpha}f(x)\in C(0,1]$ for $k=0,1,\ldots,m$. Then we have

$$f(x)=\sum_{i=0}^{m-1}\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}D^{i\alpha}f(0^+)+\frac{x^{m\alpha}}{\Gamma(m\alpha+1)}D^{m\alpha}f(\xi)\tag{3}$$

with $0<\xi\leqslant x,\ \forall x\in(0,1]$. Also, one has

$$\left|f(x)-\sum_{i=0}^{m-1}\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}D^{i\alpha}f(0^+)\right|\leqslant M_\alpha\frac{x^{m\alpha}}{\Gamma(m\alpha+1)},\tag{4}$$

where $M_\alpha\geqslant|D^{m\alpha}f(\xi)|$.

In case of $\alpha=1$, the generalized Taylor's formula (3) reduces to the classical Taylors formula.