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Automatic finite element formulation and assembly of hyperelastic higher order structural models

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ABSTRACT

The formulation of higher order structural models and their discretization using the finite element method is difficult owing to their complexity, especially in the presence of nonlinearities. In this work a new algorithm for automating the formulation and assembly of hyperelastic higher-order structural finite elements is developed. A hierarchic series of kinematic models is proposed for modeling structures with special geometries and the algorithm is formulated to automate the study of this class of higher order structural models. The algorithm developed in this work sidesteps the need for an explicit derivation of the governing equations for the individual kinematic modes. Using a novel procedure involving a nodal degree-of-freedom based automatic assembly algorithm, automatic differentiation and higher dimensional quadrature, the relevant finite element matrices are directly computed from the variational statement of elasticity and the higher order kinematic model. Another significant feature of the proposed algorithm is that natural boundary conditions are implicitly handled for arbitrary higher order kinematic models. The validity algorithm is illustrated with examples involving linear elasticity and hyperelasticity.

1. Introduction

The analysis of partial differential equations using the finite element method comprises of a series of steps: formulation of the governing equations, discretization of the geometry, discretization of variables by specification of element-wise shape functions, finite element assembly and post-processing of the solution to obtain the desired quantities of interest. Each of these individual steps can be automated to varying extents, typically resulting in a higher level finite element framework that improves the overall efficiency of the analysis. The formulation of finite element stiffness matrices in each element of the discretized domain is typically carried out using a known algebraic form of the constituent terms. This process can be automated to various levels and a variety of algorithms that partially or fully automate the finite element method have been developed in the past [1–3]. This work deals with the automation of the finite element formulation and assembly of higher order structural/waveguide models and extends an earlier work by the authors [4] that deals with the automation of finite element analysis in the context of energy–momentum conserving integrators for 1D waveguides.

Automatic differentiation [5–7], also known as algorithmic differentiation, is used in this work for the exact calculation of derivatives that arise in the finite element formulation. Automatic differentiation takes advantage of the internal representation of functions in the computer memory as compositions of basic functions and algebraic operators, and the chain rule of differentiation [6]. Only the forward mode of automatic differentiation [6] is used in this work. It is implemented as a C++ class with appropriate operator overloading that provides facility for handling *dual numbers* and automatic differentiation. While automatic differentiation is used to numerically calculate the derivatives of the required functions that arise in the finite

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element computations, a significant feature of the proposed algorithm is to extend it to automate the formulation of the tangents stiffness matrix in nonlinear finite element problems, without using an explicit algebraic form of the individual terms. Thus the algorithm presented in this work also automates the linearization of nonlinear finite element variational forms.

A good approximation to the response of structures with special geometries like rods, beams, plates etc., can be obtained by choosing appropriate kinematic models that take advantage of the special geometric features of the structure under consideration [8–15]. These higher order models reduce the 3D problem to a lower dimensional equivalent and thus significantly reduce the problem size. They provide a very good approximation to the actual response as long as the geometry of the structure satisfies certain constraints and thus serve as an efficient alternative to a full 3D analysis.

The accuracy of a higher order structural model is directly related to how well the actual deformation is captured by the higher order kinematic modes employed in the model. While the introduction of kinematic modes reduces the problem to a lower dimensional equivalent, the use of a large number of modes makes it cumbersome to formulate the governing equations for the structural response. Models with very few modes are sufficient to study linear response under small loads, but they may not be sufficient when nonlinearities are present and/or when large loads are applied. The necessity of higher order modes is especially important for studying the high frequency response of structures [12,16]. However, with higher order models the governing equations take a very complicated form [11,12,17]. For nonlinear problems, in addition to the difficulty of deriving the governing equations for the individual kinematic modes, the calculation of the tangent stiffness matrix, which involves the linearization of the governing equations, becomes very cumbersome. Computer algebra systems may not be efficient as the number of modes increases and it is even more difficult to cast them in a finite element framework. To alleviate these difficulties, a class of hierarchic higher order kinematic models and a new algorithm that automates the finite element formulation and assembly of these higher-order structural models are developed in this work. The algorithm automates the formulation the tangent stiffness matrices directly from a given kinematic model without requiring an algebraic derivation of the governing equations for the individual modes.

The algorithm is developed within a general variational framework and is thus applicable to both linear and nonlinear problems. Another advantage of using a variational framework is that the natural boundary conditions are implicitly taken care of in the formulation [13,14]. The difficulty of treating natural boundary conditions becomes pronounced for higher order kinematic models even in the linear case. For nonlinear problems it is very difficult to satisfy the natural boundary conditions, primarily because kinematic models impose restrictions on the variation of the displacement variables and hence do not directly deal with the stresses. Hence the stress–strain equations need to be solved for satisfying the natural boundary conditions. While this is possible in the case of linear elasticity, it is not feasible for general nonlinear models. In this work, this problem is automatically handled for arbitrary higher order kinematic models and the natural boundary conditions are satisfied, to the accuracy of the kinematic approximation, without any need to derive them separately as would be required in a conventional formulation.

Higher order kinematic models typically use polynomials in the lateral (thickness) coordinate variables to model the kinematic modes. For linear problems these terms typically factor out as the area or the moment of inertia. For higher order nonlinear structural problems it is cumbersome to calculate all these factors algebraically. In the present method, this problem is eliminated based on the observation that these factors, typically lower order polynomials, can be exactly evaluated with the use of appropriate numerical quadratures over the cross-section, thus avoiding the need for an explicit derivation of their algebraic form. Thus numerical integration is performed in all three dimensions even for a 1D or 2D waveguide model. This use of higher dimensional quadrature for solving a lower dimensional structural model is another unique feature of the proposed algorithm.

The three main themes introduced above – an algorithm for automatic finite element assembly of a given variational form, automatic differentiation and a novel procedure to integrate arbitrary higher order structural models directly into the variational framework – are developed and synthesized into a new and flexible procedure for studying higher order structural finite elements. The validity of the proposed procedure is illustrated with a variety of examples. While full geometric nonlinearity is considered in this work, only the hyperelastic form of material nonlinearity is demonstrated in the analysis. The procedure however is quite general and can be extended for other types of material nonlinearities.

The outline of the paper is as follows: the governing equations of elasticity and hyperelasticity are summarized first. A hierarchic class of higher order structural models that are subsequently used for illustrating the automatic algorithm is then introduced. A brief discussion of automatic differentiation and the basic algorithm for automatic finite element assembly is presented next in the context of a general variational form. The algorithm is then extended for the case of higher order structural elements. The exposition given here generalizes the earlier work by the authors [4] and some of the underlying concepts are presented again for clarity and completeness. The key advantages of the algorithm are seen when it is applied to higher order structural models, especially in the nonlinear case. This is the crux of the current work and the full procedure which combines the advantages of the automatic assembly algorithm and automatic differentiation for higher order nonlinear structural models is finally discussed with a few illustrative examples.

2. Governing equations of elasticity

The governing equations of linear elasticity and hyperelasticity in cartesian coordinates are summarized in this section to the extent they are required. The Murnaghan form of hyperelastic behaviour is introduced as a special nonlinear material