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# Self-adjoint singularly perturbed second-order two-point boundary value problems: A patching approach

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ABSTRACT

The basic aim of this study is to introduce and describe a patching approach based on a novel combination of the variational iterative method (VIM) and adaptive cubic spline collocation scheme for the solution of a class of self-adjoint singularly perturbed second-order two-point boundary value problems that model various engineering problems. The domain of the problem is decomposed into two subintervals: the VIM is implemented in the vicinity of the boundary layer while in the outer region the resulting problem is tackled by applying an adaptive cubic spline collocation scheme (ASS), which comprises the use of mapping/transformation redistribution functions or constructed grading functions.

Numerical results, computational comparisons, appropriate error measures and illustrations are provided to testify the convergence, efficiency and applicability of the method. Performance of this method is examined through test examples that reveal that the current approach converges to the exact solution rapidly and with $\mathcal{O}(h^4)$ accuracy and that the convergence is uniform across the domain. The proposed technique yields numerical solutions in very good agreement with and/or superior to existing exact and approximate solutions.


## 1. Introduction

The main purpose of this article is to present and describe a numerical patching technique based on the VIM and a fourth-order adaptive cubic spline collocation scheme for the numerical solution of the subsequent class of self-adjoint singularly perturbed second-order two-point boundary value problems.

$$L_y \equiv -\epsilon(a(x)y')' + b(x)y = g(x), \tag{1}$$

defined on $[0,1]$ and satisfying the boundary conditions:

$$y(0) = \alpha \quad y(1) = \beta, \tag{2}$$

where $\epsilon$ is a small positive parameter and $a(x), b(x)$ and $g(x)$ are smooth functions. In addition, we enforce the following conditions on these coefficients to warrantee that the operator $L_y$ satisfy a maximum principle [1]:

$$a(x) \geqslant a^* > 0, \quad a'(x) \geqslant 0, \quad b(x) \geqslant b^* > 0.$$

The class of perturbed problems 1,2 occurs in science and engineering applications, such as, quantum mechanics, fluid mechanics, chemical-reactor theory, aerodynamics, optimal control, reaction–diffusion process, and geophysics. This class is a broadly studied model that has been discussed reasonably extensively in various papers that appear in the literature.

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From the numerical point of view, many schemes for this model have been presented and analyzed. The majority of the existing numerical approaches are based on finite element methods (see [2,3]) or uniformly second-order accurate and three-point difference schemes (see [4–6]), B-spline collocation (see [7,8]), finite difference method with variable mesh (see [9]), and a designed wavelet optimized finite difference scheme (see [10]).

In recent years the VIM has been the focus of numerous articles which is utilized for attaining exact and/or numerical solutions for a wide range of nonlinear equations including algebraic, differential, partial-differential, functional-delay and integro-differential equations (see [11–13] and the references therein). The approach is efficient and handles the differential equations without imposing restrictive assumptions that may change the physical structure of the solutions. The wide-ranging and successful implementation of the VIM has demonstrated and confirmed that the method is a powerful mathematical tool for treating an assortment of linear and nonlinear problems. The main thrust of the technique method is based on constructing a correction functional using a general Lagrange multiplier, which is selected in a proper way that its correction solution is improved with respect to the initial approximation or to the trial function. The resulting solution is in the form of a successive approximations that often converge to the exact solution or its Taylor's series expansion.

On the other hand, the cubic B-spline finite element collocation approach is widely used for the numerical solution of a broad range of nonlinear problems [14] arising in various applications (see [15–18] and the references therein. In particular and besides the VIM, we will also manipulate an adaptive grid technique [19,20], based on cubic B-spline collocation on non-uniform Shishkin-like meshes (see [21–23]). This technique necessitates the redistribution of the nodes in order to have more points placed in regions of large variation of the solution, for instance those close to layers or near singularities. That is, the mesh is constructed in an proper way so that it is finer near the boundary layer or the singularity but coarser otherwise. The cubic spline collocation method is integrated with the adaptive technique ASS to solve the problem on nonuniform meshes via mapping uniform node points to non-uniform ones such that the errors are reduced and are uniformly distributed.

The main goal of the present article is to suggest a numerical patching method for tackling the model problem 1,2. The approach is based on a combination of the VIM and ASS. Certain deficiencies of the VIM and the difficulty of the art of choosing the nonuniform mesh using an appropriate mapping function or grading functions when applying the cubic spline collocation methods have motivated the investigation of a patching method that combines both. The VIM, which converges rather fast locally and is very accurate near the boundary layer, is applied in a small neighborhood about the layer. The setback of the VIM is that it the convergence deteriorates quite noticeably as the applicable domain increases. That is, it yields only a local accurate approximation using only few iterations while many more iterates are needed for values away from the origin but that might result in computational challenges and an increase in rounding error. In contrast, the cubic spline collocation provides global estimation of the solution, however, the challenge and/or complexity of the scheme is that an effort and a skill is necessary for the selection of mesh in order to obtain satisfactory approximation especially in the presence of singularity or boundary layer. To surmount the slow convergence concern of the VIM away from the origin, we use a fourth-order ASS to approximate the solution away from the origin.

The efficiency and accuracy of the numerical scheme is assessed on specific test problems. The numerical outcomes indicate that the method yields highly accurate results. The numerical solutions are compared with analytical and other existing numerical solutions in the literature. The convergence analysis is discussed and it is verified that the method has a fourth-order rate of convergence using the double-mesh principle.

The balance of this paper is organized as follows. In Section 2, the numerical patching method is presented and described for the numerical solution of the class of self-adjoint singularly perturbed equations. In Section 3, a number of test problems are discussed to assess the accuracy of the technique. The last Section 4 includes a conclusion that briefly summarizes the numerical outcomes.

## 2. Numerical method

In this section, we describe the patching method and present the ASS and VIM strategies.

### 2.1. Variational method procedure

We begin by summarizing the variational iterative method as it applies to the singular perturbation problem 1,2. The correction functional for Eq. (1) is given by

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\left[-\epsilon\big(a(s)(y_n)_s\big)_s + b(s)\tilde{y}_n - g(s)\right] ds, \quad n = 0, 1, 2, \ldots, \tag{3}$$

where $\tilde{y}_n$ is a restricted variation ($\delta\tilde{y}_n = 0$).

Next we need to find the optimal value of $\lambda(s)$. To achieve that, we first operate the variation with respect to $y_n(x)$ on both sides of the latter equation. We have

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\left(\int_0^x \lambda(s)\left[-\epsilon\big(a(s)(y_n)_s\big)_s + b(s)\tilde{y}_n - g(s)\right] ds\right) \tag{4}$$