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# Model reduction by extended minimal degree optimal Hankel norm approximation

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ABSTRACT

This paper presents an extended minimal degree optimal Hankel norm approximation (MDOHNA) based order reduction algorithm using a basis-free descriptor which circumvents the requirement of computing balanced realized model for order reduction. Conjunction of system decomposition algorithm (Singh and Nagar, 2004) [21], (Kumar et al., 2012) [30,31] with MDOHNA is used as extension for order reduction of unstable systems. The developed algorithm is applicable for stable/unstable, linear time invariant, minimal/non-minimal, continuous/discrete-time systems as well. Further, effectiveness of the algorithm over existing techniques is validated with the help of a numerical example.



## 1. Introduction

Modeling strategies often result in dynamical systems of very high dimension. It is then desirable to find systems of the same form but of lower complexity, whose input–output behavior approximates the behavior of the original system. From a mathematical and system theoretical point of view, reduction using optimal Hankel norm approximation is among the most important model reduction techniques today. It is one of the few model reduction algorithms that produce optimal approximate models.

Adamjan et al. [1] introduced a closed-form optimal solution for model reduction with respect to Hankel norm criterion for the scalar (single input–output) case. The relevance of [1] to model reduction was first mentioned by Kung [2] who later presented closed-form optimal Hankel norm solution for multivariable systems [3] and developed minimal degree approximation algorithm (MDA). The structure of linear dynamical systems with finite dimension is exploited in [4] to derive explicit algorithms and simple expressions for the Hankel norm approximation of a high dimensional discrete-time stable scalar system by a reduced model of any low order. The continuous-time scalar case is independently solved by Bettayeb et al. [5]. Kung and Genin [6] used a two-variable polynomial approach to rederive the results of [l] and described many significant properties of the MDA problems. Further Kung et al. [7] presented state space formulation of optimal Hankel norm approximation problem. Subsequently, Glover [8] investigated characterization of all optimal Hankel norm approximations that minimize the Hankel norm for multivariable linear systems and derived the frequency response error bound. Characterization of all solutions to the suboptimal Hankel problem using a different approach was also derived by Ball et al. [9,10] continuous-time and discrete-time systems. Safonov and Chiang [11] suggested an improved representation of the equations for minimal degree approximation Hankel norm model reduction which completely circumvents the need of balanced state

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space realization, permits the same simple formulas to be used for both the suboptimal and the optimal cases. Further, order reduction for different class of systems using optimal Hankel norm approximation is investigated by several authors [12–18].

As far as reduction of unstable systems is concerned, several researchers adopted different approaches. The common approach is to split the system into a stable subsystem and a unstable subsystem, and then apply an existing reduction technique to reduce the stable part while entirely retaining the unstable part. Hsu and Hou [19] considered the reduction of continuous-time systems in which the transformed system is directly used for model reduction without complete decoupling of system into stable and unstable subsystems. By using bilinear transformation, Chen [20] suggested a method for decomposition into three parts- stable, oscillatory and unstable subsystems.

In this paper extension of minimal degree optimal Hankel norm approximation (MDOHNA) technique for order reduction of unstable systems using system decomposition algorithm is proposed. The algorithm used for system decomposition [21–23] is based on real Schur transformation [24] and it is free from bilinear transformation in such a way that the original unstable system is decomposed into stable and unstable subsystems. The reduced order model is obtained by simplifying the stable subsystem and adding it to the unstable subsystem.

The organization of the paper is as follows: Section 2 describes the various steps of the MDOHNA algorithm for order reduction of stable, minimal/non-minimal, continuous/discrete-time systems. Section 3 investigates the extension of the algorithm for unstable systems using system decomposition. Finally, the comparative study of proposed method and conclusions are made in Section 4 and 5, respectively.

## 2. Order reduction algorithm by minimal degree optimal Hankel norm approximation

The development of the optimal Hankel norm approximation [1–3] and the balanced truncation [25] changed the perception of model reduction techniques significantly. These two techniques ensured almost perfect characteristics as their reduced order models are stable and also a priori frequency response error bounds. For the systems having uncontrollable and unobservable states, the balancing transforms are generally singular. This creates practical difficulties while applying standard Hankel norm approximation theory as balanced state space model is required to be computed first. The minimal degree optimal Hankel norm approximation (MDOHNA) algorithm [11] completely alleviates the requirement of computing balanced realized model for the system to be reduced, even when the original system is nearly uncontrollable and/or unobservable.

Consider the transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ and the associated standard realization of a linear time invariant (LTI) dynamical system as,

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1a}$$

$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where $A \in \Re^{n\times n},\ B \in \Re^{n\times q},\ C \in \Re^{p\times n}$ and $D \in \Re^{p\times q}$. The number of state variables $n$ is known as the order of the system. We are interested in computing a reduced-order LTI system,

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}\hat{u}(t), \tag{2a}$$

$$\hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}\hat{u}(t), \tag{2b}$$

of order $r$, $r << n$, such that the transfer function matrix $\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D}$, approximates original system $G(s)$.

The Hankel norm represents the energy transferred from past inputs to the future outputs through the system $G = \{A, B, C, D\} \in H_\infty$. If the input $u(t) = 0$ for $t \geqslant 0$, and the output is $y(t)$, then Hankel norm [8] is defined as:

$$\|G(s)\|_H = \sup_{u\in L^2(-\infty,0)} \frac{\|y\|_{L^2(0,\infty)}}{\|u\|_{L^2(-\infty,0)}}. \tag{3}$$

The associated Hankel operator [11] can be represented as,

$$H_G : L_2(-\infty, 0] \to L_2(0, \infty] : (H_G u)(t) = \int_{-\infty}^{0} G(t-\tau)u(\tau)\,d\tau. \tag{4}$$

Bettayeb et al. [5] showed that $H_G$ has singular value decomposition which can be determined directly from state space realization of $G(s)$ with any order $n \geqslant m$, where $m$ is the McMillan degree of $G(s)$. Further, $m$ non-zero Hankel singular values (HSV) of the system can be obtained by finding the square root of the eigenvalues of product of $P$ and $Q$.

$$HSV = \sqrt{\lambda_i(PQ)} \text{ such as } \sigma_1 \geqslant \sigma_2 \geqslant .... \geqslant \sigma_m > 0, \tag{5}$$

where $\lambda_i(M)$ denotes the $i$th eigenvalue of $M$ and $P$ and $Q$ are controllability and observability Gramians which can be defined as,

$$P = \int_0^\infty e^{\tau A}BB^* e^{\tau A^*}\,d\tau, \tag{6}$$