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Continuous dependence theorems on solutions of uncertain differential equations

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ABSTRACT

In ordinary differential equation (ODE) and stochastic differential equation (SDE), the solution continuously depends on initial value and parameter under some conditions. This paper investigates the analogous continuous dependence theorems in uncertain differential equation (UDE). It proves two continuous dependence theorems, a basic one and a general one.

1. Introduction

Nondeterministic phenomenon in dynamic system, such as perturbation and white noise, is usually described by random variable. As a result, stochastic differential equation becomes the main means to research dynamic system with perturbation. However, lots of surveys show that sometimes it is not suitable to regard the perturbation as random variable. For example, the data describing the dynamic system may come from expert, especially in researching economic or social system, and we can not regard the expert data as random variable. This means that SDE can not model all dynamic systems.

How can we deal with dynamic systems with expert data? In order to solve this problem, Liu proposed uncertainty theory [1,2] and uncertain differential equation [3]. Up to now, a lot of research on UDE has been started. Chen and Liu [4] and Gao [5] proved a series existence and uniqueness theorems on UDE. Yao and Chen [6] gave a numerical method to solve UDE. Zhu [7] introduced UDE into optimal control. Some researchers employed UDE to model financial market, such as Peng and Yao [8] and Chen [9].

This paper studies how the solution of UDE depends on initial value and parameter in coefficient. In ODE [10,11] and SDE [12–14], the solution is continuous with respected to initial value and parameter under some conditions. Will the analogous conclusion hold in UDE? This paper gives an affirmative answer. It first proves a basic continuous theorem, which says if the coefficients of UDE satisfy global Lipschitz condition and linear growth condition, the solution is continuously depending on initial value and parameter.

Although the basic continuous theorem is easy to understand, the global Lipschitz condition is so strict that most UDEs can not satisfy it. To solve this problem, this paper further proves a general theorem. The strong theorem only requires that the coefficients are continuous and the solution is unique.

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The remainder of this paper is organized as follows. In Section 2, some basic concepts and properties of uncertainty theory and UDE used throughout this paper are introduced. In Section 3, the basic continuous theorem is proved. Section 4 proves the general continuous theorem. Section 5 gives a brief summary to this paper.

2. Preliminary concepts and definitions

In this section, we introduce some foundational concepts and properties of uncertainty theory and UDE, which are used throughout this paper.

Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element $\Lambda \in \mathrm{L}$ is assigned a number $\mathcal{M}\{\Lambda\} \in [0,1]$. In order to ensure that the number $\mathcal{M}\{\Lambda\}$ has certain mathematical properties, Liu ([1,2]) presented the four following axioms: (1) normality, (2) self-duality, (3) countable subadditivity, and (4) product measure axioms. If the first three axioms are satisfied, the set function $\mathcal{M}\{\Lambda\}$ is called an uncertain measure.

Definition 1 (Liu [1]). Let Γ be a nonempty set, $\mathcal{L}$ a σ-algebra over Γ, and $\mathcal{M}$ an uncertain measure. Then the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

Definition 2 (Liu [1]). An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\},$$

is an event.

Definition 3 (Liu [3]). Let T be an index set and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process is a measurable function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set B of real numbers, the set

$$\{\xi_t \in B\} = \{\gamma \in \Gamma | \xi_t(\gamma) \in B\},$$

is an event.

That is, an uncertain process $X_t(\gamma)$ is a function of two variables such that the function $X_{t^*}(\gamma)$ is an uncertain variable for each t^*.

Definition 4 (Liu [15]). An uncertain process C_t is said to be a canonical process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2.

An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(x) = \mathcal{M}\{\xi \leqslant x\} = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \Re,$$

whose expected value is e and variance is σ^2.

Definition 5 (Liu [3]). Suppose C_t is a canonical process, and f and g are some given functions. Then

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{1}$$

is called an uncertain differential equation. A solution is an uncertain process X_t that satisfies (1) identically in t.

For the sake of simplicity, our discussion is not based on the form of UDE (1) but on its equivalent integral form, or say, uncertain integral equation

$$X_t = X_0 + \int_0^t f(s, X_s)\mathrm{d}s + \int_0^t g(s, X_s)\mathrm{d}C_s. \tag{2}$$

In this paper, we investigate the general form of uncertain integral Eq. (2)

$$X_t^{(p)} = X_0^{(p)} + \int_0^t f(s, X_s^{(p)}, p)\mathrm{d}s + \int_0^t g(s, X_s^{(p)}, p)\mathrm{d}C_s, \tag{3}$$

that is, the initial value $X_0^{(p)}$ and coefficients $f(t,x,p)$ and $g(t,x,p)$ all depend on parameter p. Our goal is to find how the solution $X_t^{(p)}$ depends on parameter p on a finite interval $[0, T]$.