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A stabilized finite element method for the time-dependent Stokes equations based on Crank–Nicolson Scheme ☆

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ABSTRACT

A stabilized finite element method for the time-dependent Stokes equations based on Crank–Nicolson scheme is considered in this paper. The method combines the Crank–Nicolson scheme with a stabilized finite element method which uses the lowest equal-order element pair, i.e., the stabilized finite element method is applied for the spatial approximation and the time discretization is based on the Crank–Nicolson scheme. Moreover, we present optimal error estimates and prove that the scheme is unconditionally stable and convergent. Finally, numerical tests confirm the theoretical results of the presented method.

1. Introduction

Let Ω be a bounded, convex and open subset of $\mathbb{R}^2$ with a Lipschitz continuous boundary $\partial\Omega$. We consider the time-dependent Stokes problem

$$\begin{aligned} & u_t - \nu\Delta u + \nabla p = f \quad \text{in } \Omega \times (0,T], \\ & \text{div} u = 0 \quad \text{in } \Omega \times (0,T], \\ & u(x,0) = u_0(x) \quad \text{on } \Omega \times \{0\}, \\ & u = 0 \quad \text{on } \partial\Omega \times (0,T], \end{aligned} \tag{1}$$

where $u = (u_1(x,t), u_2(x,t))$ represents the velocity vector, $p = p(x,t)$ the pressure, $f = f(x,t)$ the prescribed force, $\nu > 0$ the viscosity, $u_0(x)$ the initial velocity, $T > 0$ the given final time and $u_t = \partial u/\partial t$.

It is well known that numerical approximation of nonstationary Stokes equations plays an important role in the analysis of incompressible flow problems. Thus, development of an efficient and effective computational method for investigating nonstationary Stokes problem has practical significance, and has drawn the attention of many researchers. At the time of writing, numerous works are devoted to this problem (see [1–15], and the references cited therein).

To the author's knowledge, there exist fully implicit, semi-implicit (semi-explicit), and explicit scheme to deal with the time-dependent problem. Among them, high-order schemes are of more interest because first-order schemes are not sufficiently accurate for large time approximations. Meanwhile, the stability condition of schemes is also a key issue. Usually an explicit scheme is much easier in computation. But it suffers a severely restricted time step size from stability requirement. A

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fully implicit scheme is (almost) unconditionally stable. However, at each time step, one has to solve a system of nonlinear equations. Hence, a popular approach is based on an implicit scheme for the linear term and a semi-implicit scheme or an explicit scheme for the nonlinear term. A semi-implicit scheme for the nonlinear term results in a linear system with a variable coefficient matrix of time, and an explicit treatment for the nonlinear term gives a constant matrix. Crank–Nicolson scheme [16] is firstly proposed by Crank and Nicolson for the heat-conduction equation in 1947, and it is unconditionally stable with second-order accuracy. Moreover, because of its high accuracy and unconditional stability, the scheme has been widely used in many PDEs [17,18] and drawn the attention of many researchers for Navier–Stokes equations [19–23]. He and Sun [19] have provided an error analysis for the Crank–Nicolson extrapolation scheme of time discretization applied to the spatially discrete stabilized finite element approximation of the two-dimensional time-dependent Navier–Stokes problem, and the low-order finite element have been applied. Meanwhile, they have studied the stability and convergence of the Crank–Nicolson/Adams–Bashforth scheme for the two-dimensional non-stationary Navier–Stokes equations in [20]. They have used the conforming element, which satisfies the inf–sup condition. A fully implicit Crank–Nicolson scheme (implicit for both linear and nonlinear terms) has been proposed by Heywood and Rannacher [18], and they have proved that it is almost unconditionally stable and convergent.

Mixed finite element method is a natural choice for solving fluid mechanics equations because these equations naturally appear in mixed form in terms of velocity and pressure. In the analysis and practice of employing mixed finite element methods in solving the Stokes equations, the inf–sup condition has played an important role because it ensures a stability and accuracy of the underlying numerical schemes. A pair of finite element spaces that are used to approximate the velocity and the pressure unknowns are said to be stable if they satisfy the inf–sup condition. Intuitively speaking, the inf–sup condition is something that enforces a certain correlation between two finite element spaces so that they both have the required properties when employed for the Stokes equations. However, due to computational convenience and efficiency in practice, some mixed finite element pairs which do not satisfy the inf–sup condition are also popular. Thus, much attention has been paid to the study of the stabilized method for the Stokes problem.

Recent studies have focused on stabilization of the lowest equal-order finite element pair $P_1 - P_1$ (linear functions on triangular and tetrahedron elements), $Q_1 - Q_1$ (bilinear functions), or $P_1 - P_1$ (linear functions on quadrilateral and hexahedral elements, using the projection of the pressure onto the piecewise constant space [10,12,24]. This stabilization technique is free of stabilization parameters and does not require any calculation of high-order derivatives or edge-based data structures. Therefore, this method is gaining more and more popularity in computational fluid dynamics.

This paper focuses on the Crank–Nicolson scheme with a stabilized finite element approximation in spatial direction for solving the time-dependent Stokes equations. It is different from [19] due to that they use different stabilization methods to stabilize different elements for different equations. The main work of this paper is to use the pressure projection technique to stabilize the lowest-order conforming $P_1 - P_1$ element (i.e. adding to the bilinear form the difference between an exact Gaussian quadrature rule for quadratic polynomials and an exact Gaussian quadrature rule for linear polynomials to offset the inf–sup condition). The remainder of this paper is organized as follows. In the next section, an abstract functional setting of the Stokes problem is given and some well-known results throughout this paper are introduced. Then a stabilized mixed finite element method is reviewed in Section 3. In Section 4, we describe the Crank–Nicolson scheme and prove stability result and error estimates of this scheme. Then in Section 5 numerical experiments are shown to verify the theoretical results completely. Finally, we end with some short conclusions in Section 6.

2. Preliminaries

We shall introduce the following Hilbert spaces:

$$X = H_0^1(\Omega)^2, \quad Y = L^2(\Omega)^2, \quad M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q \mathrm{d}x = 0 \right\}.$$

The spaces $L^2(\Omega)^m,\ m = 1, 2$, are equipped with the L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $\|\cdot\|_0$. The space X is endowed with the usual scalar product $(\nabla u, \nabla v)$ and the norm $\|\nabla u\|_0$. Standard definitions are used for the Sobolev spaces $W^{m,p}(\Omega)$, with the norm $\|\cdot\|_{m,p}$, $m, p \geqslant 0$. We will write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $\|\cdot\|_m$ for $\|\cdot\|_{m,2}$. Next, let the closed subset V of X be given by

$$V = \{ v \in X : \mathrm{div} v = 0 \}$$

and denote by H the closed subset of Y, i.e.,

$$H = \{ v \in Y : \mathrm{div} v = 0,\ v \cdot n|_{\partial\Omega} = 0 \}.$$

We denote the Stokes operator by $A = -P\Delta$, where P is the L^2-orthogonal projection of Y onto H and set $D(A) = H^2(\Omega)^2 \cap V$.

We usually make the following assumptions on the prescribed data for problem (1) provided in [18,19,25].

(**A1**). Assume that Ω is smooth so that the unique solution $(v, q) \in X \times M$ of the steady Stokes problem

$$-\Delta v + \nabla q = g \quad \text{in } \Omega, \quad \mathrm{div} v = 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega$$

for any prescribed $g \in Y$ exists and satisfies

$$\|v\|_2 + \|q\|_1 \leqslant c_0 \|g\|_0, \tag{2}$$